1 Blow-up

1.1 Universal property of blow-up and resolution of singularities for curves

Last time we have seen how is possible to compute the resolution of singularities of some geometric object over a perfect field using blow up. Today we will construct the blow up of a general (noetherian) scheme. The main idea of blow up is quite clear in the situation of curves. In fact a curve is smooth if and only if is normal if and only if every ideal is locally free. So, to obtain a resolution of singularities of a curve, we have to take some sheaf of ideals and force it to be locally free of rank one.

Remark 1. Every scheme will be noetherian. If $f : X \to Y$ is a map of scheme and I is a sheaf of ideals of Y, we will denote $f^{-1}I$ the sheaf of ideal of X made by the image of $f^*I \to \mathcal{O}_X$

Definition 1. Let X be a scheme and I a sheaf of ideals. The Blow-up (if it exists) of X in I is a scheme $\pi : Bl_I(X) \to X$ such that $\pi^{-1}I$ is locally free of rank one and it is universal between all the other scheme with this property.

Remark 2. If it exists it is unique.

Theorem 1. *$Bl_I(X)$ exists and it is proper over X.*

We will prove this theorem later. Now we will assume it and we will proof that we can obtain the resolution of singularities for curve just doing a sequence of blow-ups. We need the following easy lemma.

Lemma 1.
1. *$Bl_I(X) \to X$ is an isomorphism if and only if I is locally free of rank one.*
2. *If $f : T \to X$ is flat, then $Bl_{f^{-1}I}T = Bl_I(X) \times_X T$*
3. *If $U = X - V(I)$, then $\pi^{-1}(U) \to U$ is an iso*

Proof.
1. This is clear.
2. We verify the universal property. By flatness $f^{-1}I = f^*I$. Suppose that $g : C \to T$ is such that $g^{-1}(f^{-1}I)$ is locally free of rank one. Then the image of $g^*f^*I \to \mathcal{O}_C$ is locally free and hence there exists a unique map $C \to Bl_I(X)$ and hence a unique map $C \to Bl_I(X) \times_X T$

3. Observe that if $i : U \to X$ is the (flat) inclusion then $i^{-1}I = \mathcal{O}_U$ and hence it is locally free of rank one. Then $\pi^{-1}(U) = U \times_X Bl_I(X) = Bl_{i^{-1}(I)}(U) \simeq U$

$\square$

Theorem 2. *Let X be an integral projective curve over a field k. Define $X_0 = X$ and $\pi_i : X_i = Bl_{SingX_{i-1}}(X_{i-1}) \to X_{i-1}$ where $SingX_{i-1}$ is the ideal that define the singular locus of X_{i-1}. Then the sequence $... \to X_n \to X_{n-1} ... \to X_1 \to X$ stabilizes at some n and X_n is non singular. So X_n is a resolution of singularities of X.*

Proof. Observe that every π_i is a proper birational morphism so that every X_i is a integral projective curve and π_i is a finite morphism. Moreover for every i we have an exact sequence of sheaves $0 \to \mathcal{O}_{X_i} \to \pi_* \mathcal{O}_{X_{i+1}} \to \mathcal{F}_i \to 0$, where $\mathcal{F}_n$ is a skyscraper sheaf supported in a finite number of points. Hence we have $H^1(F_n) = 0$ so that $\chi(\mathcal{O}_{X_{i+1}}) = \chi(\mathcal{O}_{X_i}) + h^0(\mathcal{F}_i)$ and so $p_a(X_{i+1}) = p_a(X_i) - h^0(\mathcal{F}_i)$ is a decreasing sequence (here $p_a(X) = 1 - \chi(\mathcal{O}_X)$). Moreover $p_a(X_i) \geq 1 - h^0(\mathcal{O}_{X_i})$ and $H^0(\mathcal{O}_{X_i}) \subseteq \overline{k} \cap K(X_n) = \overline{k} \cap K(X)$ so that it is bounded below. Hence there exists an n such that $p_a(X_{n+1}) = p_a(X_n)$ so that $h^0(\mathcal{F}_n) = 0$. But this implies $\mathcal{F}_n = 0$ and so $\mathcal{O}_{X_n} \to \pi_* \mathcal{O}_{X_{n+1}}$ is an isomorphism and hence $X_{n+1} \to X_n$ is an isomorphism. But this implies that the singular locus of X_n it is defined by a locally free ideal and this is not possible unless X_n is non singular. $\square$

Remark 3. Suppose that A is a finitely generated algebra of dimension 1 over a field. Then the normalization $\tilde{A}$ is finite over A and every ideals of $\tilde{A}$ is locally free of rank one. If we do the same construction of the previous theorem we get, by the universal property of blow-up, a family of maps as in the following diagram:

$$\begin{tikzcd}
 & & & & & Spec(\tilde{A}) \arrow[d] \arrow[dllll] \arrow[dlll] \arrow[dll] \arrow[dl] \\
... \arrow[r] & X_n \arrow[r] & X_{n-1} \arrow[r] & ... \arrow[r] & X_1 \arrow[r] & Spec(A)
\end{tikzcd}$$

Since the normalization if finite, every map is finite so that the sequence stabilizes and $Spec(\tilde{A})$ is isomorphic to some X_n. So an explicit description of blow-up will allow us to compute the normalization of affine curves.

Proof. of theorem 1 It is enough to prove the theorem when $X = Spec(A)$ is affine (thanks to the first lemma and the universal property the blow-ups of an affine cover will glue together.

So suppose $X = Spec(A)$ and $I = (f_1, ..., f_n)$. We claim that $\mathbf{Proj}(\tilde{A}) =: \tilde{X}$, where $\tilde{A}$ is the graded algebra $\oplus_{i\geq 0} I^i$, with its canonical map $\pi : \tilde{X} \to X$ is the blow-up. We have a surjective map of graded algebras $\phi : A[x_1, ..., x_n] \to \tilde{A}$ that send x_i to t_i, where t_i is the element f_i but seen in degree one. This induced a closed embedding $\psi : \tilde{X} \to \mathbb{P}^{n-1}_X$.We will show the three properties:

- $\pi^{-1}I$ is locally free of rank one.
  In fact $\pi^{-1}I$, in terms of graded modules is just the image of the map $I \otimes_A \tilde{A} \to \tilde{A}$, i.e $\oplus_{i\geq 1} I^i$. But this, as sheaf, is isomorphic to $\tilde{A}[1]$ that it is locally free of rank one since it is $\psi^*\mathcal{O}_{\mathbb{P}^{n-1}_X}(1)$

- Existence of the map.
  Suppose that $g : T \to A$ is such that $f^{-1}I$ is locally free of rank one. Then the pullback $s_1, ..., s_n$ of $f_1, ..., f_n$ trough this map give us a map $T \to \mathbb{P}^{n-1}_X$, So we have to show that this map factorize trough the closed in immersion of $\tilde{X}$ in the projective space. Now $\tilde{X}$ is defined by the homogeneous polynomials F such that $F(f_1, ..., f_n) = 0$ but $F(s_1, ..., s_n) = F(g^*f_1, ..., g^*f_n) = 0$ and so we are done.

- Uniqueness of the map.
  A map to $\tilde{X}$ is uniquely determined by a map to the projective space and a map to the projective space is uniquely determined by an locally free sheaf of rank one and some section, so that the uniqueness is clear.

□

In the same way it is possible to prove the following:

Proposition 1. *Suppose that $f : W \to X$ is a morphism of scheme and I a sheaf of ideals over X. Then there exists a unique map $\tilde{f}$ that makes the following diagram commutative. Moreover if f is a closed immersion then $\tilde{f}$ is a closed immersion.*

$$\begin{tikzcd}
Bl_{f^{-1}I}(W) \arrow[r, dotted, "\tilde{f}"] \arrow[d] & Bl_I(X) \arrow[d] \\
W \arrow[r, "f"] & X
\end{tikzcd}$$

1.2 Concrete description and examples

Now we want to describe the local structure of the blow up. We fix a ring A, an ideal $I = (f_1, ..., f_n)$. We will denote $X = Spec(A)$, $\tilde{A}$ the graded algebra $\oplus_{i\geq 0} I^i$ and with t_i the element f_i in degree 1 in this algebra. Recall that we have a a surjective map of graded algebras $\phi : A[x_1, ..., x_n] \to \tilde{A}$ that send x_i to t_i and that $\tilde{X} := \mathbf{Proj}(\tilde{A}) = Bl_I(X)$. We cover $\mathbb{P}^{n-1}_A$ with the standard affine open subsets $A_i = Spec(A[s_{1,i}, ..., s_{n,i}]) = Spec(A[\frac{x_1}{x_i}, ..., \frac{x_n}{x_i}])$ that is the homogeneous localization of $A[x_1, ..., x_n]$ in the prime ideal (x_i)

Lemma 2. *$Ker(\phi)_{(x_i)} = \{P \in A[s_{1,i}, ..., s_{n,i}]$ such that $\exists d \geq 0$ $f_i^d P(S) \in ((f_i s_{j,i} - f_j)_j)\} := J_i$*

Proof. If $P \in A[s_{1,i}, ..., s_{n,i}]$, there exists a natural number d such that $f_i^d P(S) = \sum_{j\neq i} Q_j(f_i s_{j,i} - f_j) + a$.
If $P \in Ker(\phi)_{(x_i)}$ then $0 = f_i^d P(\frac{t_1}{t_i}, ..., \frac{t_n}{t_i}) = \sum_{j\neq i} Q_j(\frac{t_1}{t_i}, ..., \frac{t_n}{t_i})(f_i \frac{t_j}{t_i} - f_j) + a = a$ so that there exists a r such that $t_i^r a = 0$ and hence $f_i^r a = 0$. Now $f_i^{d+r} P(S) \in ((f_i s_{j,i} - f_j)_j)$ and we are done.
For the reverse inclusion, if $P \in J_i$ write P as $\frac{Q(X)}{x_i^e}$ for some e and some homogeneous polynomial of degree e. The $f_i^d x_i^e Q(X)$ is in $((f_i x_j - f_j x_i)_j)$ and hence $f_1^d t_i^e Q(t_1, ..., t_n) = 0$ so that $f_1^d t_i^e Q(t_1, ..., t_n) = 0$. In conclusion $t_i^{e+d} Q(t_1, ..., t_n) = 0$ and hence $P(S) = \frac{Q(T)}{x_i^{r+d+e}} x_i^{d+e}$ is in the kernel. □

Proposition 2. *$\tilde{X}$ Is covered by open affine isomorphic to $Spec(A_i)$, where A_i is the sub algebra of A_{f_i} generated by $\frac{f_j}{f_i}$.*

Proof. $\tilde{X}$ is covered by the $Spec(\tilde{A}_{(t_i)})$ and $\tilde{A}_{(t_i)} \simeq \frac{A[s_{1,i}, ..., s_{n,i}]}{J_i}$. To conclude observe that the map $A[s_{1,i}, ..., s_{n,i}] \to A_{f_i}$ that send $s_{j,i}$ to $\frac{f_j}{f_i}$ has kernel J_i and the sub algebra of A_{f_i} generated by $\frac{f_j}{f_i}$ as image. □

Now we can recover the classical description of the blow up.

Corollary 1. *Suppose that the f_i are a minimal system of generators and that $Z = V(J)$ is irreducible in $\mathbb{P}^{n-1}$, where J is the ideal generated by the elements $f_i x_j - f_j x_i$. Then the closed immersion of $\tilde{X}$ in $\mathbb{P}^{n-1}$ induce an isomorphism $\tilde{X} \simeq V(J)$*

Proof. Clearly $\tilde{X} \subseteq V(J)$ and so to check that the map is an isomorphism we can check on $U_i = D_+(x_i) \cap V(J)$. Observe that, by minimality, f_i is non zero in $\mathcal{O}_Z(U_i)$ so it not a zero divisor (by irreducibility) and hence we are done by the lemma. □

Before doing some example we'd like the understand what is the fibers over $V(I)$.

Proposition 3. *Suppose that I is generated by a regular sequence $f_1, ..., f_n$, then $\pi^{-1}(V(I)) \simeq \mathbb{P}^{n-1}_{A/I}$*

Proof. We have $\pi^{-1}(V(I)) \simeq (\mathrm{Proj} \oplus_i I^i) \times_A Spec(A/I) = \mathrm{Proj} \oplus_i I^i/I^{i+1}$ so we have only to show that $\oplus_i I^i/I^{i+1} \simeq Sym((A/I)^{n-1})$.
Observe that the Kozsul complex $K_A(f_1, ..., f_n)$ is exact in degree bigger then 0. Then we have a morphism $B := \mathbb{Z}[x_1, ..., x_n] \to A$ that send x_i to f_i. Since $x_1, ..., x_n$ is a regular sequence, $K_B(x_1, ..., x_n)$ is an acyclic resolution of B/J, where $J = (x_1, ..., x_n)$, and observe that $K_A(f_1, ..., f_n) = K_B(x_1, ..., x_n) \otimes_B A$. Then $Tor^q_B(A, B/J)$ can be computed as the cohomology of the complex $K_A(f_1, ..., f_n)$ that it is exact in degree bigger then 0 and hence $Tor^q_B(A, B/J) = 0$ if $q > 0$.
Now observe that J^i/J^{i+1} is a free B/J module generated by $x_1^{k_1}...x_n^{k_n}$ with $\sum k_i = i$ and hence $J^i/J^{i+1} \simeq (B/J)^{n_k}$ for some n_k so that $Tor^q_B(A, J^i/J^{i+1}) = 0$. By induction, using the exact sequence $0 \to J^{k-1}/J^{k'} \to J^k/J^{k'} \to J^{k-1}/J^k \to 0$, one gets that $Tor^q_B(A, J^k/J^{k'}) = 0$ for every $k' > k$ and $q > 0$. Thanks to this, using the exact sequence $0 \to J^k \to B \to B/J^k \to 0$, we have that $I^k/I^{k+1} \simeq J^k/J^{k+1} \otimes_B A \simeq J^k/J^{k+1} \otimes_{B/J} B/J \otimes_B A \simeq J^k/J^{k+1} \otimes_{B/J} A/I$. So we get that I^k/I^{k+1} is a A/I free module generated by $f_1^{k_1}...f_n^{k_n}$ with $\sum k_i = k$ and hence that $\oplus_i I^i/I^{i+1} \simeq Sym((A/I)^{n-1})$. $\square$

Corollary 2. *Suppose that $V(I)$ is a regular subscheme of dimension n. Then for every $x \in V(I)$, $\pi^{-1}(x) \simeq \mathbb{P}^{n-1}_{k(x)}$*

Proof. Use the previous proposition and the flat base change for blow-up. $\square$

Now finally we get can back the usual blow-up of the plane at the origin.

Example 1. Take $A = \mathbb{A}^n_k$ and $I = (x_1, ..., x_n)$. Then, by prop 1, $Bl_I(X)$ is isomorphic to the closed subscheme of $\mathbb{P}^{n-1}_A = \mathbb{P}^{n-1}_k \times A^n_k$ defined by $(x_i t_j - x_j t_i)$. π is an isomorphism outside the origin and, thanks to the proposition 3, the fiber of π in the origin is $\mathbb{P}^{n-1}_k$. It is possible to show that $Bl_I(X) = Spec(Sym(\mathcal{O}_{\mathbb{P}^{n-1}_k}(1))$. It is a regular scheme.

Example 2. Consider $A = \frac{k[x,y]}{(y^2-x^3)}$ and $I = (x, y)$. By proposition 2, it is covered by two open affine isomorphic to the spectrum of $A_1 = \frac{k[x,y]}{(y^2-x^3)}[\frac{x}{y}] = k[\frac{x}{y}, \frac{y}{x}]$ and $A_2 = \frac{k[x,y]}{(y^2-x^3)}[\frac{y}{x}] = k[\frac{y}{x}]$ and hence it is regular. Moreover $A_2 \subseteq A_1$ so that $Bl_I(X) \simeq \mathbb{A}^2_k$. Observe that, despite the fact that (x, y) is not a regular point, the fiber of π over it it is just one point.

Example 3. Consider $A = \frac{k[x,y]}{(xy)}$ and $I = (x, y)$. One of its open affine is isomorphic to the spectrum of A_1, the A subalgebra of $A_x = k[x]_x$ generated by $\frac{y}{x} = 0$ and hence it is isomorphic to $\mathbb{A}^1_k$. The same computation show that $Spec(A_2)$ is isomorphic to $\mathbb{A}^1_k$. Since A is not irreducible, $Bl_I(X)$ is isomorphic to the disjoint union of two lines. Observe that the fiber over the origin is made by two point, so that it is not isomorphic to $\mathbb{P}^0_k$.

Example 4. As last example consider $A = \frac{k[x,y,z]}{(z^2-xy)}$ and $I = (x, y, z)$. The blow up is covered by

- $\frac{k[x,y,z]}{(z^2-xy)}[\frac{y}{x}, \frac{z}{x}] = k[\frac{z}{x}, x]$
- $\frac{k[x,y,z]}{(z^2-xy)}[\frac{x}{y}, \frac{z}{y}] = k[\frac{z}{y}, y]$
- $\frac{k[x,y,z]}{(z^2-xy)}[\frac{x}{z}, \frac{y}{z}] = k[\frac{y}{z}, z]_{\frac{y}{z}}$

and hence it is regular.