# A NOTE ON THE BEHAVIOUR OF THE TATE CONJECTURE UNDER FINITELY GENERATED FIELD EXTENSIONS

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ABSTRACT. We show that the $\ell$-adic Tate conjecture for divisors on smooth proper varieties over finitely generated fields of positive characteristic follows from the $\ell$-adic Tate conjecture for divisors on smooth projective surfaces over finite fields. Similar results for cycles of higher codimension are given.

## 1. INTRODUCTION

Let $k$ be a field of characteristic $p \geq 0$ with algebraic closure $\overline{k}$ and write $\pi_1(k)$ for the absolute Galois group of $k$. A $k$-variety is a reduced scheme, separated and of finite type over $k$. For a $k$-variety $Z$ write $Z_{\overline{k}} := Z \times_k \overline{k}$ and $CH^i(Z_{\overline{k}})$ for the group of algebraic cycles of codimension $i$ modulo rational equivalence. Let $\ell \neq p$ be a prime.

1.1. **Conjectures.** Recall the following versions of the Grothendieck-Serre-Tate conjectures ([Tat65], [And04, Section 7.3]):

**Conjecture 1.1.1.** If $k$ is finitely generated and $Z$ is a smooth proper $k$-variety, then:

- $T(Z, i, \ell)$ : The $\ell$-adic cycle class map
$$c_{Z_{\overline{k}}} : CH^i(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \to \bigcup_{[k':k]<+\infty} H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k')}$$
is surjective;
- $S(Z, i, \ell)$ : The action of $\pi_1(k)$ on $H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))$ is semisimple;
- $WS(Z, i, \ell)$ : The inclusion $H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k)} \subseteq H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))$ admits a $\pi_1(k)$-equivariant splitting.

For a field $K$, one says that $T(K, i, \ell)$ holds if for every finite field extension $K \subseteq L$ and every smooth proper $L$-variety $Z$, $T(Z, i, \ell)$ holds. One defines similarly the conditions $S(K, i, \ell)$ and $WS(K, i, \ell)$.

Conjecture 1.1.1 is widely open in general. By the works of many people, $T(Z, 1, \ell)$ is known when $Z$ is an abelian variety ([Tat66], [Zar75], [Zar77], [FW84]), a K3 surface ([NO85], [Tan95], [And96a], [Cha13], [MP15], [KMP15]) or when $Z$ lies in some other special class of $k$-varieties; see for example [MP15, Section 5.13] and [Moo17].

### 1.2. Behaviour under finitely generated field extensions.

1.2.1. *Main result.* For abelian varieties and K3 surfaces, Conjecture 1.1.1 is closely related to the finiteness of rational points on their moduli spaces; see [Tat66, Proposition 2] and [LMS14]. This may suggest that Conjecture 1.1.1 could be easier to prove when $k$ is a finite field. The main result of this note is that, to prove Conjecture 1.1.1 for varieties over finitely generated fields of positive characteristic, it is actually enough to prove it for varieties over finite fields.

**Theorem 1.2.1.** *If $p > 0$, then $T(\mathbb{F}_p, i, \ell)$ and $WS(\mathbb{F}_p, i, \ell)$ imply $T(k, i, \ell)$ for every finitely generated field $k$ of characteristic $p$.*

1.2.2. *The case of divisors.* By [Tat94, Proposition 2.6.], if algebraic and numerical equivalences on algebraic cycles coincide rationally in degree $i$, then $T(Z, i, \ell)$ implies $WS(Z, i, \ell)$. Since this holds for divisors, from Theorem 1.2.1 we deduce the following.

**Corollary 1.2.2.** *If $p > 0$, then $T(\mathbb{F}_p, 1, \ell)$ implies $T(k, 1, \ell)$ for every finitely generated field $k$ of characteristic $p$.*

By an unpublished result ([dJ]) of De Jong (whose proof has been simplified in [Mor15, Theorem 4.3]), over finite fields the $\ell$-adic Tate conjecture for divisors for smooth projective varieties follows from the $\ell$-adic Tate conjecture for divisors for smooth projective surfaces. Hence Corollary 1.2.2 implies the following:

**Corollary 1.2.3.** *If $p > 0$, then $T(Z, 1, \ell)$ for every finite field $\mathbb{F}_q$ of characteristic $p$ and every smooth projective $\mathbb{F}_q$-surface $Z$ implies $T(k, 1, \ell)$ for every finitely generated field $k$ of characteristic $p$.*

1.3. **Previously known results.** We quickly review previously known results on the behaviour of Conjecture 1.1.1 under finitely generated field extension. Let $k \subseteq K$ be a regular finitely generated field extension and let $Z$ be a smooth proper $K$-variety. Choose a geometrically connected, smooth $k$-variety $\mathcal{K}$ with generic point $\eta : K \to \mathcal{K}$ and a smooth proper morphism $\mathfrak{f} : \mathcal{Z} \to \mathcal{K}$ fitting into a cartesian diagram:

$$\begin{tikzcd} Z \arrow[r, "i_\eta"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[d, "\mathfrak{f}"] \\ k \arrow[r, "\eta"] & \mathcal{K}. \end{tikzcd}$$

For every $\mathfrak{t} \in \mathcal{K}$, choose a geometric point $\bar{\mathfrak{t}}$ over $\mathfrak{t}$ and write $\mathcal{Z}_{\mathfrak{t}}$ and $\mathcal{Z}_{\bar{\mathfrak{t}}}$ for the fibre of $\mathfrak{f}$ at $\mathfrak{t}$ and $\bar{\mathfrak{t}}$ respectively.

1.3.1. *Characteristic zero.* Assume that $k$ has characteristic zero. Then the arguments in [And96b, Section 5.1], show that $S(k, i, \ell)$ and $T(k, i, \ell)$ imply $T(K, i, \ell)$. The idea is use resolution of singularities to embed $\mathcal{Z}$ as a dense open subset into a smooth proper $k$-variety $\mathcal{Z}^{cmp}$. Then, by smooth proper base change the action of $\pi_1(K)$ on $H^{2i}(Z_{\overline{K}}, \mathbb{Q}_\ell(i))$ factors through the surjection $\pi_1(K) \twoheadrightarrow \pi_1(\mathcal{K})$. By the global invariant cycles theorem ([Del80]; see [And06, Theoreme 1.1.1]), the natural map $H^{2i}(\mathcal{Z}^{cmp}_{\overline{k}}, \mathbb{Q}_\ell(i)) \to H^{2i}(Z_{\overline{K}}, \mathbb{Q}_\ell(i))^{\pi_1(\mathcal{K}_{\overline{k}})}$ is surjective. Since $S(\mathcal{Z}^{cmp}_{\overline{k}}, i, \ell)$ holds, the map

$$H^{2i}(\mathcal{Z}^{cmp}_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k)} \to H^{2i}(Z_{\overline{K}}, \mathbb{Q}_\ell(i))^{\pi_1(K)}$$

is still surjective. Hence, by $T(\mathcal{Z}^{cmp}, i, \ell)$, every class in $H^{2i}(Z_{\overline{K}}, \mathbb{Q}_\ell(i))^{\pi_1(K)}$ arises from a cycle in $H^{2i}(\mathcal{Z}^{cmp}_{\overline{k}}, \mathbb{Q}_\ell(i))$.

Let us point out that, even assuming $S(\mathbb{F}_p, i, \ell)$ instead of $WS(\mathbb{F}_p, i, \ell)$ in Theorem 1.2.1, the arguments in [And96b, Section 5.1] do not work in positive characteristic, since resolution of singularities is not known. On the other hand our arguments for Theorem 1.2.1 do not work in characteristic zero, since they use in an essential way the procylicity of $\pi_1(\mathbb{F}_q)$.

1.3.2. *Semisimplicity.* As observed in [Fu99], Deligne's geometric semisimplicity theorem [Del80, Theoreme 3.4.1] can be used to show that $S(k, i, \ell)$ implies $S(K, i, \ell)$. More precisely, by Deligne's geometric semisimplicity theorem ([Del80, Theoreme 3.4.1]), the restriction of the action of $\pi_1(\mathcal{K})$ on $H^{2i}(Z_{\overline{K}}, \mathbb{Q}_\ell(i))$ to its normal subgroup $\pi_1(\mathcal{K}_{\overline{k}}) \subseteq \pi_1(\mathcal{K})$ is semisimple. For any closed point $\mathfrak{t} \in \mathcal{K}$, the subgroup of $\pi_1(\mathcal{K})$ generated by $\pi_1(\mathcal{K}_{\overline{k}})$ and by the image of $\pi_1(\mathfrak{t}) \to \pi_1(\mathcal{K})$ is open in $\pi_1(\mathcal{K})$. So, since the action of $\pi_1(\mathcal{K}_{\overline{k}})$ is semisimple, the action of $\pi_1(\mathcal{K})$ on $H^{2i}(Z_{\overline{K}}, \mathbb{Q}_\ell(i))$ is semisimple if the action of $\pi_1(\mathfrak{t})$ on $H^{2i}(Z_{\overline{K}}, \mathbb{Q}_\ell(i))$ induced via restriction through the morphism $\pi_1(\mathfrak{t}) \to \pi_1(\mathcal{K})$ is semisimple. But this action identifies, modulo the isomorphism $H^{2i}(Z_{\overline{K}}, \mathbb{Q}_\ell(i)) \simeq H^{2i}(\mathcal{Z}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(i))$ given by the choice of an étale path between $\overline{\eta} : \overline{K} \to \mathcal{K}$ and $\bar{\mathfrak{t}}$, with the natural action of $\pi_1(\mathfrak{t})$ on $H^{2i}(\mathcal{Z}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(i))$. Hence $S(\mathcal{Z}_{\mathfrak{t}}, i, \ell)$ implies $S(Z, i, \ell)$.

1.3.3. *Infinite finitely generated fields.* Assume now that $k$ is **infinite** and finitely generated. Then the results in [And96b] (see [Cad12, Corollary 5.4]) if $p = 0$ or [Amb18, Theorem 1.3.3] if $p > 0$, show that $T(k, 1, \ell)$ implies $T(K, 1, \ell)$. Indeed, they show that there exists always a closed fibre $\mathcal{Z}_{\mathfrak{t}}$ such that the Néron-Severi group $NS(\mathcal{Z}_{\mathfrak{t}}) \otimes \mathbb{Q}$ of $\mathcal{Z}_{\mathfrak{t}}$ identifies (rationally) with the Néron-Severi group $NS(Z) \otimes \mathbb{Q}$ of $Z$. Since the choice of an étale path between $\bar{\mathfrak{t}}$ and $\overline{\eta}$ induces a commutative diagram of injective maps

$$\begin{tikzcd} NS(Z) \otimes \mathbb{Q}_\ell \arrow[r, hook, "\simeq"] \arrow[d, hook] & NS(\mathcal{Z}_{\mathfrak{t}}) \otimes \mathbb{Q}_\ell \arrow[d, hook] \\ H^2(Z_{\overline{K}}, \mathbb{Q}_\ell)^{\pi_1(K)} \arrow[r, hook] & H^2(\mathcal{Z}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(1))^{\pi_1(\mathfrak{t})}, \end{tikzcd}$$

this shows that $T(\mathcal{Z}_{\mathfrak{t}}, 1, \ell)$ implies $T(Z, 1, \ell)$.

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## 2. Proof of Theorem 1.2.1

Let $k$ be an infinite finitely generated field $k$ of characteristic $p > 0$ inside a fixed algebraic closure $\overline{k}$. Let $\mathbb{F}_q$ (resp. $\mathbb{F}$) the algebraic closure of $\mathbb{F}_p$ in $k$ (resp. $\overline{k}$). For every smooth proper $k$-variety $Z$, write $CH^i_\ell(Z_{\overline{k}})$ for the image of $CH^i(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))$. Let $Z$ be a smooth and proper $k$-variety.

2.1. **Strategy.** Compared to the arguments Section in 1.3.1, the extra difficulties one has to deal with in the proof of Theorem 1.2.1 come from the fact that resolution of singularities is not known in positive characteristic and that we don't assume the semisimplicity of the Galois action in $\ell$-adic cohomology.

To overcome the use of resolution of singularities, we use De Jong's alterations theorem to construct a generically étale alteration $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ such that $\widetilde{\mathcal{Z}}$ embeds as a dense open subset into a smooth proper $\mathbb{F}_q$-variety. As pointed out by a referee, the resulting morphism $\widetilde{\mathcal{Z}} \to \mathcal{Z} \to \mathcal{K}$ is not, in general, generically smooth, so that we cannot apply directly the global invariant cycles theorem. To solve this issue, we use the main ingredients of its proof: the Hard Lefschetz theorem [Del80, Theorem 4.1.1] and the theory of weights for $\mathbb{F}_q$-varieties [Del80, Theorem 1].

To overcome the lack of the semisimplicity assumption we combine the procyclicty of $\pi_1(\mathbb{F}_q)$ with the condition $WS(Z, i, \ell)$, to study the fixed points of the action of $\pi_1(\mathbb{F}_q)$ (Section 2.3.4) via the generalized eigenspace of generalized eigenvalue 1 of a topological generator.

2.2. **Preliminary reductions.** To prove $T(Z, i, \ell)$, one may freely replace $k$ with a finite field extension. In particular we may assume that all the connected components of $Z_{\overline{k}}$ are defined over $k$ and so, working with each component separately, that $Z$ is geometrically connected over $k$. The following well known lemma, a slight variant of [Tat94, Theorem 5.2], will be used twice.

**Lemma 2.2.1.** *Let $W$ be a smooth proper $k$-variety and $g : W \to Z$ a generically finite dominant morphism. Then the following hold:*

- *The map $g^* : H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i)) \to H^{2i}(W_{\overline{k}}, \mathbb{Q}_\ell(i))$ is injective.*
- *For any $z \in H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))$, if $g^*(z) \in CH^i_\ell(W_{\overline{k}})$ then $z \in CH^i_\ell(Z_{\overline{k}})$.*

*In particular $T(W, i, \ell)$ implies $T(Z, i, \ell)$.*

*Proof.* Assume first that $W$ is geometrically connected. Then, by Poincaré duality, there is a morphism $g_* : H^{2i}(W_{\overline{k}}, \mathbb{Q}_\ell(i)) \to H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))$ which is compatible with the push forward of cycles $g_* : CH^i(W_{\overline{k}}) \otimes \mathbb{Q}_\ell \to CH^i(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell$ and such that $g_* g^*$ is equal to the multiplication by the generic degree of $g : W \to Z$. All the assertions then follow from the commutative diagram:

$$\begin{tikzcd}
CH^i(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[r, "g^*"] \arrow[d, "c_{Z_{\overline{k}}}"] & CH^i(W_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[r, "g_*"] \arrow[d, "c_{W_{\overline{k}}}"] & CH^i(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[d, "c_{Z_{\overline{k}}}"] \\
H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i)) \arrow[r, "g^*"] & H^{2i}(W_{\overline{k}}, \mathbb{Q}_\ell(i)) \arrow[r, "g_*"] & H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i)).
\end{tikzcd}$$

In general, we reduce to the situation where $W$ is geometrically connected. To prove Lemma 2.2.1, we can freely replace $k$ with a finite field extension and hence assume that all the connected components $W_{i,\overline{k}}$ of $W_{\overline{k}}$ are defined over $k$. Since $g : W \to Z$ is dominant and generically finite and $Z$ is connected, there is at least one connected component (say $W_1$) mapping surjectively onto $Z$. Since $Z$ and $W_1$ are smooth proper $k$-varieties of the same dimension, the morphism $g_1 : W_1 \to W \to Z$ is still dominant and generically finite. The general case follows then from the geometrically connected case and the diagram:

$$\begin{array}{ccccc} CH^i(Z_{\overline{k}})\otimes\mathbb{Q}_\ell & \longrightarrow & CH^i(W_{\overline{k}})\otimes\mathbb{Q}_\ell & \longrightarrow & CH^i(W_{1,\overline{k}})\otimes\mathbb{Q}_\ell \\ \downarrow & & \downarrow & & \downarrow \\ H^{2i}(Z_{\overline{k}},\mathbb{Q}_\ell(i)) & \longrightarrow & H^{2i}(W_{\overline{k}},\mathbb{Q}_\ell(i)) & \longrightarrow & H^{2i}(W_{1,\overline{k}},\mathbb{Q}_\ell(i)). \quad \square \end{array}$$

By De Jong's alteration theorem ([dJ96]) applied to $Z_{\overline{k}}$, there exists a smooth projective $\overline{k}$-variety $W'$ and a dominant generically finite morphism $g' : W' \to Z_{\overline{k}}$. By descent and replacing $k$ with a finite field extension, there exist a smooth projective $k$-variety $W$ and a dominant generically finite morphism $g : W \to Z$ which, after base change along $Spec(\overline{k}) \to Spec(k)$, identifies with $g' : W' \to Z_{\overline{k}}$. By Lemma 2.2.1, we may replace $Z$ with $W$ and hence we may assume that $Z$ is a smooth projective $k$-variety. Since the action of $\pi_1(k)$ on $CH^i_\ell(Z_{\overline{k}})$ factors through a finite quotient, replacing $k$ with a finite field extension, we may and do assume that $CH^i_\ell(Z_{\overline{k}}) \subseteq H^{2i}(Z_{\overline{k}},\mathbb{Q}_\ell(i))^{\pi_1(k)}$. The core of the proof is the following proposition.

**Proposition 2.2.2.** *Let $Z$ be a geometrically connected smooth projective $k$-variety such that $CH^i_\ell(Z_{\overline{k}}) \subseteq H^{2i}(Z_{\overline{k}},\mathbb{Q}_\ell(i))^{\pi_1(k)}$. Assume that $T(\mathbb{F}_p,i,\ell)$ holds. Replacing $k$ with a finite field extension, there exist a projective $k$-scheme $\widetilde{Z}$ and a dominant generically finite morphism $h : \widetilde{Z} \to Z$, such that for every $z \in H^{2i}(Z_{\overline{k}},\mathbb{Q}_\ell(i))^{\pi_1(k)}$ one has $h^*(z) \in CH^i_\ell(\widetilde{Z}_{\overline{k}})$.*

Before proving Proposition 2.2.2, let us show that it implies Theorem 1.2.1. Replacing $k$ with a finite field extension we can take $h : \widetilde{Z} \to Z$ as in the statement of Proposition 2.2.2. Write $\widetilde{Z}_{\overline{k},red}$ for the reduced closed subscheme of $\widetilde{Z}_{\overline{k}}$. Then $h_{red} : \widetilde{Z}_{\overline{k},red} \to \widetilde{Z}_{\overline{k}} \to Z_{\overline{k}}$ is still dominant and generically finite and for every $z \in H^{2i}(\widetilde{Z}_{\overline{k}},\mathbb{Q}_\ell(i))^{\pi_1(k)}$ one has $h^*_{red}(z) \in CH^i_\ell(\widetilde{Z}_{\overline{k},red})$. So, by descent and replacing $k$ with a finite field extension, we can assume that $\widetilde{Z}$ is geometrically reduced and that all the irreducible components of $\widetilde{Z}_{\overline{k}}$ are defined over $k$. Then, by De Jong alteration's theorem applied to $\widetilde{Z}_{\overline{k}}$ and descent, replacing $k$ with a finite field extension, there exists a generically finite dominant morphism $W \to \widetilde{Z}$ with $W$ a smooth projective $k$-variety. The morphism $g : W \to \widetilde{Z} \to Z$ is still generically finite and dominant and for every $z \in H^{2i}(Z_{\overline{k}},\mathbb{Q}_\ell(i))^{\pi_1(k)}$ one has $g^*(z) \in CH^i_\ell(W_{\overline{k}})$. Then Theorem 1.2.1 follows from Lemma 2.2.1.

The next subsection is devoted to the proof of Proposition 2.2.2.

2.3. **Proof of Proposition 2.2.2.** Let $Z$ be a geometrically connected smooth projective $k$-variety such that $CH^i_\ell(Z_{\overline{k}}) \subseteq H^{2i}(Z_{\overline{k}},\mathbb{Q}_\ell(i))^{\pi_1(k)}$.

2.3.1. *Spreading out and alterations.* Spreading out to $\mathbb{F}_q$, there exist a geometrically connected, smooth $\mathbb{F}_q$-variety $\mathcal{K}$ with generic point $\eta : k \to \mathcal{K}$ and a smooth projective morphism $\mathfrak{f} : \mathcal{Z} \to \mathcal{K}$ with geometrically connected fibres fitting into a cartesian diagram:

$$\begin{array}{ccc} Z & \xrightarrow{i_\eta} & \mathcal{Z} \\ \downarrow & \square & \downarrow \mathfrak{f} \\ k & \xrightarrow{\eta} & \mathcal{K}. \end{array}$$

By De Jong alteration's theorem, there exist an integral smooth $\mathbb{F}_q$-variety $\widetilde{\mathcal{Z}}$, an open embedding $\widetilde{\mathfrak{i}} : \widetilde{\mathcal{Z}} \to \widetilde{\mathcal{Z}}^{cmp}$ with dense image into a smooth projective $\mathbb{F}_q$-variety $\widetilde{\mathcal{Z}}^{cmp}$ and a generically étale, proper, dominant morphism $\mathfrak{h} : \widetilde{\mathcal{Z}} \to \mathcal{Z}$. Then $\widetilde{\mathcal{Z}}^{cmp}$ is geometrically connected over some finite field extension $\mathbb{F}_q \subseteq \mathbb{F}_{q'}$. Replacing $\mathbb{F}_q$ with $\mathbb{F}_{q'}$ amounts to replacing $k$ with the finite field extension $k' := k\mathbb{F}_{q'}$, so we can assume that $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{Z}}^{cmp}$ are geometrically connected over $\mathbb{F}_q$.

Since $\widetilde{\mathcal{Z}} \to \mathcal{Z} \to \mathbb{F}_q$ is quasi-projective, the morphism $\mathfrak{h} : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is quasi-projective as well ([SP, Tag 0C4N]). Since $\mathfrak{f} : \mathcal{Z} \to \mathcal{K}$ is projective, this implies that $\widetilde{\mathcal{Z}} \to \mathcal{K}$ is quasi-projective. Since $\mathfrak{h} : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ and $\mathfrak{f} : \mathcal{Z} \to \mathcal{K}$ are proper, the morphism $\widetilde{\mathcal{Z}} \to \mathcal{K}$ is proper as well. So $\widetilde{\mathcal{Z}} \to \mathcal{K}$ is proper and quasi-projective hence projective. The generic fibre $\widetilde{Z} \to k$ of $\widetilde{\mathcal{Z}} \to \mathcal{K}$ is then a projective $k$-scheme endowed with a generically finite dominant morphism $h : \widetilde{Z} \to Z$. The situation is summarized in the following diagram of $\mathbb{F}_q$-schemes:

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, "\widetilde{i}_\eta"] \arrow[d, "h"] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{Z}} \arrow[r, hook, "\widetilde{\mathfrak{i}}"] \arrow[d, "\mathfrak{h}"] & \widetilde{\mathcal{Z}}^{cmp} \\
Z \arrow[r, "i_\eta"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[d, "\mathfrak{f}"] & \\
k \arrow[r, "\eta"] & \mathcal{K}. &
\end{tikzcd}$$

The edge map

$$\mathrm{Ler} : H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \to H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell(i))$$

in the Leray spectral sequence for $\mathfrak{f}_{\mathbb{F}} : \mathcal{Z}_{\mathbb{F}} \to \mathcal{K}_{\mathbb{F}}$ fits then into a commutative diagram:

$$\begin{tikzcd}
 & & & CH^i(\widetilde{Z}_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[d, "c_{\widetilde{Z}_{\overline{k}}}"] & CH^i(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[l, "h^*"'] \arrow[d, "c_{Z_{\overline{k}}}"] \\
CH^i(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}) \otimes \mathbb{Q}_\ell \arrow[r, "\widetilde{\mathfrak{i}}^*"] \arrow[d, "c_{\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}}"] & CH^i(\widetilde{\mathcal{Z}}_{\mathbb{F}}) \otimes \mathbb{Q}_\ell \arrow[urr, "\widetilde{i}_\eta^*"] \arrow[d, "c_{\widetilde{\mathcal{Z}}_{\mathbb{F}}}"] & CH^i(\mathcal{Z}_{\mathbb{F}}) \otimes \mathbb{Q}_\ell \arrow[l, "\mathfrak{h}^*"] \arrow[urr, "i_\eta^*"'] \arrow[d, "c_{\mathcal{Z}_{\mathbb{F}}}"] & H^{2i}(\widetilde{Z}_{\overline{k}}, \mathbb{Q}_\ell(i)) & H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i)) \arrow[l, "h^*"'] \\
H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \arrow[r, "\widetilde{\mathfrak{i}}^*"] & H^{2i}(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \arrow[urr, "\widetilde{i}_\eta^*"'] & H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \arrow[l, "\mathfrak{h}^*"] \arrow[urr, "i_\eta^*"'] \arrow[rr, "\mathrm{Ler}"] & & H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell(i)). \arrow[u, hook]
\end{tikzcd}$$

2.3.2. *Hard Lefschetz Theorem.* Write $\varphi \in \pi_1(\mathbb{F}_q)$ for the arithmetic Frobenius of $\mathbb{F}_q$ and, for every $\pi_1(\mathbb{F}_q)$-representation $V$, write $V^{\varphi}_{gen}$ for the generalized eigenspace on which $\varphi$ acts with generalized eigenvalue $1$. Since $\pi_1(\mathbb{F}_q)$ is procyclic the $\pi_1(\mathbb{F}_q)$-equivariant inclusion $V^{\varphi}_{gen} \subseteq V$ as a $\pi_1(\mathbb{F}_q)$-equivariant splitting. Hence if $r : V \to W$ is $\pi_1(\mathbb{F}_q)$-equivariant morphism of continuous $\pi_1(\mathbb{F}_q)$-representations, one has

$$\mathrm{Im}(r : V^{\varphi}_{gen} \to W) = \mathrm{Im}(r : V \to W) \cap W^{\varphi}_{gen}. \tag{2.3.2.1}$$

Let $z$ be in $H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k)}$. In this section we lift $h^*(z) \in H^{2i}(\widetilde{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i))$ to $H^{2i}(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\varphi}_{gen}$. By smooth proper base change, the action of $\pi_1(k)$ on $H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))$ factors through the canonical surjection $\pi_1(\mathcal{K}) \to \pi_1(k)$, hence $H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k)} \simeq H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(\mathcal{K})}$. Consider the diagram:

$$H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \xrightarrow{\widetilde{\mathfrak{i}}^*} H^{2i}(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \xleftarrow{\mathfrak{h}^*} H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \xrightarrow{\mathrm{Ler}} H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell(i)).$$

Since

$$z \in H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k)} \simeq H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell(i))^{\pi_1(\mathbb{F}_q)} \subseteq H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell(i)),$$

the element $z$ is in $H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell(i))^{\varphi}$. Recall the following consequence of the Hard Lefschetz Theorem:

**Fact 2.3.2.2.** *The map*

$$\mathrm{Ler} : H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \to H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell(i))$$

*is surjective*

*Proof.* Let $\mathcal{F} \in Pic(\mathcal{Z}_{\mathbb{F}})$ be a relative ample line bundle for $\mathfrak{f}_{\mathbb{F}} : \mathcal{Z}_{\mathbb{F}} \to \mathcal{K}_{\mathbb{F}}$, write $m$ for the relative dimension of $\mathfrak{f}_{\mathbb{F}} : \mathcal{Z}_{\mathbb{F}} \to \mathcal{K}_{\mathbb{F}}$ and, to simplify the notation, choose an identification $\mathbb{Q}_\ell(i) \simeq \mathbb{Q}_\ell$. For every integer $r \geq 1$, $c_{\mathcal{Z}_{\mathbb{F}}}(\mathcal{F}) \in H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell)$ induces, by cup product, a morphism

$$u^r : R^{m-r}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell \to R^{m+r}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell.$$

By proper base change and the Hard Lefschetz Theorem [Del80, Theorem 4.1.1], $u^r : R^{m-r}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell \to R^{m+r}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell$ is an isomorphism. In the terminology of [Del68, (2.2)], this shows that $\mathbb{Q}_\ell$ satisfies the Lefschetz condition with respect to $c_{\mathcal{Z}_{\mathbb{F}}}(\mathcal{F})$. By [Del68, Proposition 2.4], this implies that the Leray spectral sequence for $\mathfrak{f}_{\mathbb{F}} : \mathcal{Z}_{\mathbb{F}} \to \mathcal{K}_{\mathbb{F}}$

$$E_2^{a,b} := H^a(\mathcal{Z}_{\mathbb{F}}, R^b\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell) \Rightarrow H^{a+b}(\mathcal{K}_{\mathbb{F}}, \mathbb{Q}_\ell)$$

degenerates at the second page. Hence the edge map $H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell) \to H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell)$ is surjective. $\square$

By Fact 2.3.2.2, the map $\mathrm{Ler}: H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \to H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell(i))$ is surjective, hence, by (2.3.2.1), $z \in H^0(\mathcal{K}, R^{2i}\mathfrak{f}_*\mathbb{Q}_\ell(i)) \subseteq H^0(\mathcal{K}_{\mathbb{F}}, R^{2i}\mathfrak{f}_{\mathbb{F}*}\mathbb{Q}_\ell(i))^{\varphi}_{gen}$ is the image of some $z' \in H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\varphi}_{gen}$ and then $\mathfrak{h}^*(z') \in H^{2i}(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\varphi}_{gen}$.

2.3.3. *Theory of weights.* We now prove that $\mathfrak{h}^*(z')$ is the image of some $\widetilde{z} \in H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\varphi}_{gen}$ via $\widetilde{\mathfrak{i}}^* : H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \to H^{2i}(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(i))$. Write $d$ for the common dimension of $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{Z}}^{cmp}$. The localization exact sequence for the dense open immersion $\widetilde{\mathcal{Z}} \to \widetilde{\mathcal{Z}}^{cmp}$ with complement $\mathcal{D} := \widetilde{\mathcal{Z}}^{cmp} - \widetilde{\mathcal{Z}}$, gives an exact sequence

$$H^{2d-2i-1}_c(\mathcal{D}_{\mathbb{F}}, \mathbb{Q}_\ell(-i))(d) \to H^{2d-2i}_c(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(-i))(d) \to H^{2d-2i}_c(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(-i))(d).$$

Combining this sequence with Poincaré duality for the smooth varieties $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{Z}}^{cmp}$, one sees that

$$\text{(2.3.3.1)} \qquad \mathrm{Coker}\Big(\widetilde{\mathfrak{i}}^* : H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \to H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i))\Big) \subseteq (H^{2d-2i-1}_c(\mathcal{D}_{\mathbb{F}}, \mathbb{Q}_\ell(-i))(d))^\vee.$$

We combine (2.3.3.1) with the theory of weights.

**Fact 2.3.3.2** ([Del80, Théorème 3.3.1 and Corollaire 3.3.9])**.** *Let $\mathcal{X}$ be a separated scheme of finite type over $\mathbb{F}_q$. Then, for every integers $m \geq 0$ and $n$, $H^m_c(\mathcal{X}_{\mathbb{F}}, \mathbb{Q}_\ell(n))$ is mixed of weights $\leq m - 2n$. If $\mathcal{X}$ is smooth and proper over $\mathbb{F}_q$, then $H^m(\mathcal{X}_{\mathbb{F}}, \mathbb{Q}_\ell(n))$ is pure of weights $m - 2n$.*

By (2.3.3.1) and Fact 2.3.3.2, the cokernel of $\widetilde{\mathfrak{i}}^* : H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \to H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i))$ is mixed of weights $\geq 1$, while $H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i))$ is pure of weight 0. Hence, the image of $\widetilde{\mathfrak{i}}^* : H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \to H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i))$ consists exactly of the generalized eigenspace on which $\varphi$ acts with generalized eigenvalues of weight 0. So $\mathfrak{h}^*(z') \in H^{2i}(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\varphi}_{gen}$ is in the image of $\widetilde{\mathfrak{i}}^* : H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i)) \to H^{2i}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(i))$, hence by (2.3.2.1), $\mathfrak{h}^*(z')$ is the image of some $\widetilde{z} \in H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\varphi}_{gen}$.

2.3.4. *Using the Tate conjecture.* Since $WS(\widetilde{\mathcal{Z}}^{cmp}, i, \ell)$ holds by assumption, the injection

$$H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\pi_1(\mathbb{F}_q)} \hookrightarrow H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i))$$

has a $\pi_1(\mathbb{F}_q)$-equivariant splitting. So, since $\pi_1(\mathbb{F}_q)$ is procyclic generated by $\varphi$, one has

$$H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\varphi}_{gen} = H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\varphi} = H^{2i}(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(i))^{\pi_1(\mathbb{F}_q)}.$$

Hence, by $T(\widetilde{\mathcal{Z}}^{cmp}, i, \ell)$, there exists a $\widetilde{w} \in CH^i(\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}) \otimes \mathbb{Q}_\ell$ such that $c_{\widetilde{\mathcal{Z}}^{cmp}_{\mathbb{F}}}(\widetilde{w}) = \widetilde{z}$. We conclude the proof observing that, thanks to the commutative diagram at the end of 2.3.1, $h^*(z)$ is the image of $\widetilde{i}^*_{\eta}\widetilde{\mathfrak{i}}^*(\widetilde{w})$ via $c_{\widetilde{Z}_{\overline{k}}} : CH^i(\widetilde{Z}_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^{2i}(\widetilde{Z}_{\overline{k}}, \mathbb{Q}_\ell(i))$. This concludes the proof of Proposition 2.2.2.

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