REDUCTION MODULO p OF THE NOETHER'S PROBLEM

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ABSTRACT. Let k be an algebraically closed field of characteristic $p \geq 0$ and V a faithful k-rational representation of an ℓ-group G. The Noether's problem asks whether V/G is stably birational to a point. While if $\ell = p$ it is well-known that V/G is always rational, when $\ell \neq p$, Saltman and then Bogomolov constructed ℓ-groups for which V/G is not stably rational. Hence, the geometry of V/G depends heavily on the characteristic of the field. We show that for all the groups G constructed by Saltman and Bogomolov, one cannot interpolate between the Noether problem in characteristic 0 and p. More precisely, we show that it does not exist a complete valuation ring R of mixed characteristic $(0, p)$ and a smooth proper R-scheme $X \to \mathrm{Spec}(R)$ whose special fiber and generic fiber are both stably birational to V/G. The proof combines the integral p-adic Hodge theoretic results of Bhatt-Morrow-Scholze with the study of indefinitely closed differential forms in positive characteristic.

CONTENTS

1. INTRODUCTION

1.1. **Noether problem.** Let G be a finite group and let k be an algebraically closed field of characteristic $p \geq 0$. Recall that two (irreducible) k-varieties X, Y are said to be stably birational if $X \times \mathbb{P}^n_k$ is birational to $Y \times \mathbb{P}^m_k$ for some $n, m \in \mathbb{N}$. If X is stably birational to a point, we say that X is stably rational. We write $\mathrm{Stab}(k)$ for the set of varieties up to the stably birational equivalence relation and for a k-variety X we write $[X]_k \in \mathrm{Stab}(k)$ for its class in $\mathrm{Stab}(k)$.

Let $G \to GL(V)$ a faithful (finite-dimensional) representation of G. It is well known ([BK85, Lemma 1.3]) that $X := V/G$ is a normal unirational variety whose stably birational class $[G]_k$ does not depend on the chosen faithful representation of G. Knowing more about $[G]_k$ is a fundamental question in algebraic geometry that dates back to Emmy Noether ([Noe17]), when she asked whether $[G]_k = [\{\mathrm{Spec}(k)\}]_k$, which is now known as the *Noether's problem* for (G, k).

1.1.1. *Dependence on the field.* We remark that when k is not algebraically closed, there might be arithmetic obstructions to the Noether problem. For instance, Fischer theorem [Fis15] asserts that V/G is always rational when G is cyclic and $k = \overline{\mathbb{Q}}$, whereas Swan's original counterexample [Swa69] to the Noether problem uses $G = \mathbb{Z}/47\mathbb{Z}$ and $k = \mathbb{Q}$. We will focus on the case of an ℓ-group G, where ℓ is a prime number, and k is algebraically closed. In this case, the Noether's problem for G heavily depends on $p := \mathrm{char}(k)$. Indeed, while a well-known result (see e.g. [Kun54] and [Gas59]) states that $[G]_k = [\{\mathrm{Spec}(k)\}]_k$ if $\ell = p$, there are many ℓ-groups G with $\ell \neq p$ for which the Noether problem has a negative answer.

Date: February 8, 2023.

The most relevant for our purposes are the ones constructed by Saltman in [Sal84] and Bogomolov (refining Saltman's techniques) in [Bog88]. Both these examples are over $\mathbb{C}$, and use the non-triviality of the unramified Bruaer group to deduce that the quotient variety is not stably rational. Other examples using the third unramified cohomology have been constructed more recently by Peyre in [Pey08].

Since many techniques to study rationality problems are based on specialization methods both in equicharacteristic and in mixed characteristic (see e.g. [Voi15], [CTP16], [Tot16]) and the geometry of $[G]_k$ heavily depends on p, it seems interesting to understand whether one can specialize the Noether's problem for a group G from characteristic zero to positive characteristic. More precisely, let R be a complete valuation ring of mixed characteristic (0,p) with algebraically closed fraction field K and residue field k.

Definition 1.1.1. Let G be a p-group. We say that the Noether problem for G has good-reduction at p if there exists a smooth projective scheme X/R whose generic fibre has stable birational class $[G]_K$ and whose special fibre has stable birational class $[G]_k$ (and hence is stably rational) for some R as above.

When $[G]_K = [\mathrm{Spec}(K)]_K$, one can take $\mathbb{P}^n_R$ as a smooth projective model, so the question can be interesting only when $[G]_K \neq [\mathrm{Spec}(K)]_K$. As shown in Section 3, adapting the constructions in [HPT18] and following a suggestion of Colliot-Théléne, irrational varieties can have good rational reduction modulo p. Hence, understanding for which p-groups G the Noether problem has good-reduction at p is a non-trivial problem.

As a corollary of our main Theorem 1.2.2.1, we get the following non-existence result.

Corollary 1.1.2. *For all the p-groups G constructed in* [Sal84] *and in* [Bog88] *the Noether problem does not have good reduction at p.*

If one considers $\mathbb{P}^n_R$ together with a R-linear action of G such that the induced action on $\mathbb{P}^n_k$ is faithful, Corollary 1.1.2 implies that one cannot resolve the singularities of $\mathbb{P}^n_R/G$ over R in all these cases.

1.2. **Stably Birational invariants.** The main ingredient to prove Corollary 1.1.2 is the study certain stably-birational invariants. To introduced them, let X be a smooth proper variety over an algebraically closed field L.

1.2.1. *Extremal Hodge number.* By [CR11] and the Künneth formula for de-Rham cohomology, the extremal Hodge number of X

$$h^{0,i} := \dim(H^i(X, \mathcal{O}_X)) \quad h^{i,0} := \dim(H^0(X, \Omega^i_X))$$

are stably birational invariants[1]. These can be used to obstruct the existence of families of Kummer surfaces in mixed characteristic $(0,2)$ leading to results similar to Corollary 1.1.2. More precisely, let R be a complete DVR of mixed characteristic $(0,2)$ and $A \to \mathrm{Spec}(R)$ be an abelian scheme of relative dimension 2 with supersingular special fiber A_k. By [Kat78], the Kummer surface $\mathrm{Kum}(A_k)$ is a rational variety, hence $H^0(\mathrm{Kum}(A_k), \Omega^2) = 0$, while $\mathrm{Kum}(A_K)$ is a K3-surfaces, hence $H^0(\mathrm{Kum}(A_K), \Omega^2) = K$. Hence, by semicontinuity of coherent cohomology, there does not exist a smooth proper R-scheme $Y \to \mathrm{Spec}(R)$ such that $[Y_k]_k = [\mathrm{Kum}(A_k)]_k$ and $[Y_K]_K = [\mathrm{Kum}(A_K)]_K$.

Remark 1.2.1. On the positive side, let us point out that a recent result of Lazda-Skorobogatov ([LS22]) shows that if A_k is not supersingular then such a model exists. So the extremal Hodge numbers of $\mathrm{Kum}(A_k)$ are the only obstruction to the construction of a mixed characteristic family interpolating the Kummer construction.

[1]More precisely, the birational invariance of $h^{i,0}$ follows from Hartogs' lemma, while the birational invariance $h^{0,i}$ follows from the Hodge symmetry in characteristic 0 and from the main result of [CR11] in positive characteristic. To pass from birational to stable birational invariance, one uses the Künneth formuma for de-Rham cohomology.

1.2.2. *Artin-Mumford invariant.* These arguments cannot be used to study smooth proper varieties birational to V/G, since all of their extremal Hodge numbers vanish in characteristic 0.

Let $\mathrm{Br}(X)$ be the Brauer group of X and $\mathrm{Br}(X)_{div}$ the maximal divisible subgroup of $\mathrm{Br}(X)$. Let ℓ be a prime different from the characteristic of L. By the stably birational invariance of the Brauer group, also the group $H^3_{\mathrm{et}}(X, \mathbb{Z}_\ell)[\ell] \simeq \mathrm{Br}(X)/\mathrm{Br}(X)_{div}[\ell]$ is a stable birational invariant for smooth proper varieties. It is usually called the ℓ-adic Artin-Mumford $AM_\ell([X]_L)$ of $[X]_L \in SB(L)$.

The way in which Saltman (and then Bogomolov) proved that $[G]_L \neq [\{\mathrm{Spec}(L)\}]_L$, for L of characteristic 0, was by showing that $\mathrm{A}M_\ell([G]_L) \neq 0$ for his particular choice of G. Our main result is that the presence of such obstruction to stable rationality also prevents the specialization to a smooth projective, stably rational variety in characteristic p:

Theorem 1.2.2.1. *Let R be a complete valuation ring of mixed characteristic (0,p) with algebraically closed fraction field K and residue field k. Let $X \to R$ be a smooth proper R-variety whose special fiber is stably rational. Then $H^3(X_K, \mathbb{Z}_p)[p] = 0$*

In particular, $H^3(X_{\mathbb{C}}, \mathbb{Z})[p] = 0$ for any embedding $K \subset \mathbb{C}$. Corollary 1.1.2 follows then directly from Theorem 1.2.2.1 and the discussion above it.

1.3. **Integral p-adic Hodge theory.** To state the second main result of the paper, we explain our strategy to prove Theorem 1.2.2.1, of which we retain the notation. For every prime $\ell \neq p$, the smooth proper base change theorem shows that

$$H^3(X_K, \mathbb{Z}_\ell)[\ell] \simeq H^3(X_k, \mathbb{Z}_\ell)[\ell] = 0,$$

where the last equality follows from the stable birational invariance of $H^3(X_k, \mathbb{Z}_\ell)[\ell]$ and the assumption on X_k.

For p-adic coefficients, the smooth proper base change theorem does not hold except in a few special cases, but it can be replaced with the recent developments in integral p-adic Hodge theory by Bhatt-Morrow-Scholze. By [BMS18, Theorem 1.1 (ii)], one has the following inequality

$$\dim_k(H^3_{\mathrm{crys}}(X_k)[p]) \geq \dim_{\mathbb{F}_p}(H^3_{\mathrm{et}}(X_K, \mathbb{Z}_p)[p]), \tag{1.3.1}$$

where $H^3_{\mathrm{crys}}(X_k) := H^3_{\mathrm{crys}}(X_k/W)$ is the third integral crystalline cohomology group of X_k, and which will play the role of the semicontinuity theorem used in the Kummer surface examples (Section 1.2.1).

1.3.1. *Torsion in crystalline cohomology.* Thanks to (1.3.1), to prove Theorem 1.2.2.1 it would be enough to show that $\dim_k(H^3_{\mathrm{crys}}(X_k)[p])$ is a stably birational invariant. While this would follow easily from a strong form of resolution of singularities (in particular, weak factorization of birational maps) it is unclear how to prove it without assuming it. The main issue is that crystalline cohomology with integral coefficient is very badly behaved for open varieties, so the standard techniques used in characteristic zero do not work anymore.

Nevertheless, we prove something slightly weaker, but which is strong enough to imply Theorem 1.2.2.1.

Theorem 1.3.1. *Let k be an algebraically closed field of characteristic p and X a smooth proper k-variety. Assume that*

(1) $h^{0,i}_X = h^{i,0}_X = 0$ *for* $1 \leq i \leq 3$,
(2) $\mathrm{Br}(X) = 0$.

Then $H^3_{\mathrm{crys}}(X/W)[p] = 0$.

Since, as already mentioned, the assumptions of Theorem 1.3.1 are satisfied if X is stably rational (or, more generally, if X admits a decomposition of the diagonal) Theorem 1.3.1 and the inequality (1.3.1) imply Theorem 1.2.2.1.

The crucial point in the proof of Theorem 1.3.1 is to show that, under the hypothesis of the theorem, the cycles class map

$$\mathrm{Cl}_{\mathrm{dR}} : \mathrm{NS}(X) \otimes k \to H^2_{\mathrm{dR}}(X)$$

is an isomorphism, hence that a version of the Hodge conjecture for divisors holds. To prove this, the main idea to use indefinitely closed differential forms to cut out a well-behaved piece of the second de-Rham cohomology group of X. Under the assumption of Theorem 1.3.1, we then show that there are enough of these forms to generate the whole cohomology and, combining this with the theory of the de-Rham Witt complex, we get Theorem 1.3.1.

1.4. **Acknowledgement.** The authors are grateful to Stefan Schreieder for having introduced the second named author to the Noether problem, to Giuseppe Ancona for many interesting discussions and comments on a preliminary version of the article, and to Colliot-Théléne for pointing us out the example in Section 3 .

2. Proof of Theorem 1.3.1

In this section we prove Theorem 1.3.1 and so we retain the assumption and the notation therein. For $a, b, n \in \mathbb{N}$, set

$$h^{a,b} := \dim_k(H^b(X, \Omega^a_X)) \quad \text{and} \quad h^n_{\mathrm{dR}} := \dim_k(H^n_{\mathrm{dR}}(X)).$$

2.1. **Preliminary reductions.** We start observing that the universal coefficient theorem gives an exact sequence

$$0 \to H^2_{\mathrm{crys}}(X/W)/p \to H^2_{\mathrm{dR}}(X) \to H^3_{\mathrm{crys}}(X/W)[p] \to 0,$$

hence it is enough to show that the injection $H^2_{\mathrm{crys}}(X/W)/p \hookrightarrow H^2_{\mathrm{dR}}(X)$ is surjective. To prove this, we use the commutative diagram

$$\begin{array}{ccc} \mathrm{NS}(X) \otimes W & \xrightarrow{\mathrm{Cl}_{\mathrm{crys}}} & H^2_{\mathrm{crys}}(X/W) \\ \downarrow\!\!\downarrow & & \downarrow \\ \mathrm{NS}(X) \otimes k & \xrightarrow{\mathrm{Cl}_{\mathrm{dR}}} & H^2_{\mathrm{dR}}(X), \end{array} \tag{2.1.1}$$

in which the horizontal arrows are the cycle class maps for crystalline and de-Rham cohomology and which shows that it is enough to prove that

(1) $\mathrm{Cl}_{\mathrm{crys}} : \mathrm{NS}(X) \otimes W \xrightarrow{\simeq} H^2_{\mathrm{crys}}(X/W)$ is an isomorphism and
(2) $\dim_k(\mathrm{NS}(X) \otimes k) \geq h^2_{\mathrm{dR}}(X)$.

Remark 2.1.2. While it would be enough to prove directly that $\mathrm{Cl}_{\mathrm{dR}} : \mathrm{NS}(X) \otimes k \to H^2_{\mathrm{dR}}(X)$ is surjective, we cannot prove it without passing through crystalline cohomology to show that it is injective and then compare the dimensions. We remark that the equality of the dimensions of $\mathrm{NS}(X) \otimes k$ and $H^2_{\mathrm{dR}}(X)$ is not enough to guarantee that the cycle class map is surjective or injective, has shown by the example of supersingular K3 surfaces.

2.2. **Proof of (1).**

2.2.1. *Preliminaries on the De-Rham Witt complex.* Before starting with the proof of (1), we recall what we need from the theory of the de-Rham Witt complex, which will be the main tool to interpolate between the Néron-Severi group of X, its integral crystalline cohomology groups, and its flat cohomology. In [Ill79] it is defined a k-linear complex

$$W\Omega^\bullet_X : \quad W\Omega^0_X = W\mathcal{O}_X \xrightarrow{d} W\Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} W\Omega^{d-1}_X \xrightarrow{d} W\Omega^d_X$$

of quasi coherent $W\mathcal{O}_X$-modules such that

$$\mathbb{H}^n(W\Omega^\bullet_X) \simeq H^n_{\mathrm{crys}}(X),$$

see [Ill79, Theorem 1.4, Pag. 606]. If X is clear from the context, we remove the subscript and we write just $W\Omega^i$

We let $W\Omega^{\geq i}$ to be the naive truncation of the complex at i, so that there is a short exact sequence

$$0 \to W\Omega^{\geq 1} \to W\Omega^{\bullet} \to W\mathcal{O}_X \to 0 \quad (2.2.1.1)$$

of complexes. Let σ be the lift of the Frobenius of k to $W(k)$. The complex $W\Omega^{\bullet}_X$ and the sheaves $W\Omega^i_X$ are endowed with a natural σ-semilinear endomorphism[2] $\mathbf{F} : W\Omega^{\bullet}_X \to W\Omega^{\bullet}_X$, see [Ill79, Section 2.19, Pag 564]. We write $F : W\Omega^{\bullet}_X \to W\Omega^{\bullet}_X$ for the morphism which is $p^i\mathbf{F}$ is degree i and F' for the one which is $p^{i-1}\mathbf{F}$ in degree i, see [Ill79, Scholie 2.8., Pag 610] and [Ill79, Corollary 3.29, Pag 582] respectively.

By [Ill79, (5.5.2), Pag 627], there is an exact sequence

$$\cdots \to H^i_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \to H^i(X, W\Omega^{\geq 1}) \xrightarrow{1-F'} H^i(X, W\Omega^{\geq 1}) \to \dots \quad (2.2.1.2)$$

where

$$H^i_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) := \varprojlim_n H^i_{\mathrm{fl}}(X, \mu_{p^n}).$$

2.2.2. *Proof of (1).* The cycle class map $\mathrm{NS}(X) \to H^2_{\mathrm{crys}}(X)$ factors trough the morphism $\mathbb{H}^1(X, W\Omega^{\geq 1}) \to \mathbb{H}^2(X, W\Omega^{\bullet}) \simeq H^2_{\mathrm{crys}}(X)$ induced by the inclusion $W\Omega^{\geq 1} \subseteq W\Omega^{\bullet}$ and the Kummer exact sequence for the flat topology

$$0 \to \mu_{p^n} \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to 0 \quad (2.2.2.1)$$

induces ([Ill79, (5.8.5), Pag. 629]) an injection

$$\mathrm{Cl}_{\mathrm{fl}} : \mathrm{NS}(X) \otimes \mathbb{Z}_p \hookrightarrow H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$$

fitting into a commutative diagram

$$\begin{array}{ccccc} \mathrm{NS}(X) \otimes \mathbb{Z}_p & \hookrightarrow & \mathrm{NS}(X) \otimes W(k) & & \\ \downarrow \mathrm{Cl}_{\mathrm{fl}} & & \downarrow \mathrm{Cl}_{\mathrm{crys}} & \searrow^{\mathrm{Cl}_{\mathrm{crys}}} & \\ H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) & \longrightarrow & \mathbb{H}^2(X, W\Omega^{\geq 1}) & \longrightarrow & \mathbb{H}^2_{\mathrm{crys}}(X), \end{array} \quad (2.2.2.2)$$

in which the bottom left arrow is induced from the exact sequence (2.2.1.2) and bottom right one is induced by the exact sequence (2.2.1.1).

Since $0 = Br(X) \simeq H^2_{\mathrm{fl}}(X, \mathbb{G}_m)$, the exact sequence (2.2.2.1) shows that $\mathrm{Cl}_{\mathrm{fl}} : \mathrm{NS}(X) \otimes \mathbb{Z}_p \hookrightarrow H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1))$ is an isomorphism. Hence, it is enough to show that both the natural maps

$$H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \otimes W \to \mathbb{H}^2(X, W\Omega^{\geq 1}) \to \mathbb{H}^2(X, W\Omega^{\bullet})(\simeq H^2_{\mathrm{crys}}(X))$$

are isomorphisms. To be able to prove this, we will prove the following slightly stronger facts:

(a) the map $H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \otimes W \to \mathbb{H}^2(X, W\Omega^{\geq 1})$ is an isomorphism
(b) the map $\mathbb{H}^i(X, W\Omega^{\geq 1}) \to \mathbb{H}^i(X, W\Omega^{\bullet})$ is an isomorphism for $1 \leq i \leq 2$

Prove first (b) and then (a), since knowing (b) will be important to prove (a).

(b) Using the exact sequence

$$0 \to W\Omega^{\geq 1}_X \to W\Omega^{\bullet}_X \to W\mathcal{O}_X \to 0,$$

we reduce to show that $H^i(X, W\mathcal{O}_X) = 0$ for $1 \leq i \leq 2$. By [Ill79, (4.1.1), Pag 620], there exists an exact sequence

$$0 \to W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \to \mathcal{O}_X \to 0,$$

[2] In [Ill79], $\mathbf{F}$ is denoted with F, but we prefer this notation to make clearer the difference between $\mathbf{F}$ and F

where V is the Verschiebung morphism. So, since $H^i(X, \mathcal{O}_X) = 0$ for $1 \leq i \leq 3$, the morphism $V : H^i(X, W\mathcal{O}_X) \xrightarrow{V} H^i(X, W\mathcal{O}_X)$ is an isomorphism for $1 \leq i \leq 2$. On the other hand, by [Ill79, Corollaire 2.5, Pag. 609], $H^i(X, W\mathcal{O}_X)$ is topologically separated for the topology induced by V, hence, for $1 \leq i \leq 2$,

$$0 = \bigcap_n V^n(H^i(X, W\mathcal{O}_X)) = \bigcap_n H^i(X, W\mathcal{O}_X) = H^i(X, W\mathcal{O}_X).$$

(a) By point (b), the $W(k)$-module $\mathbb{H}^2(X, W\Omega^{\geq 1}) \simeq \mathbb{H}^2(X, W\Omega^\bullet)$ is finitely generated, hence by [Ill79, Lemme 6.8.4, Pag 643] and the exact sequence (2.2.1.2) it is enough to show that

$$\mathrm{Ker}(1-F') = H^2_{\mathrm{fl}}(X, \mathbb{Z}_p(1)) \quad \text{and} \quad F' : \mathbb{H}^2(X, W\Omega^{\geq 1}) \to \mathbb{H}^2(X, W\Omega^{\geq 1}) \text{ is an isomorphism.}$$

Since, by (b), also $\mathbb{H}^1(X, W\Omega^{\geq 1})$ is a finitely generated $W(k)$-module, the morphism $1 - F' : \mathbb{H}^1(X, W\Omega^{\geq 1}) \to \mathbb{H}^1(X, W\Omega^{\geq 1})$ is surjective by [Ill79, Lemma 5.3, Pag 627], so that the equality on the left follows from the exact sequence (2.2.1.2) We are left to show that $F' : \mathbb{H}^2(X, W\Omega^{\geq 1}) \to \mathbb{H}^2(X, W\Omega^{\geq 1})$ is an isomorphism. Since the natural map $\mathbb{H}^2(X, W\Omega^{\geq 1}) \to \mathbb{H}^2(X, W\Omega^\bullet)$ is an isomorphism by point (b), this would follow from [Ill79, Corollaire 5.20, Pag 636], once we verify its assumptions. To this end, we have to check that

- $H^0(X, Z^1_X) = H^0(X, \Omega^1_X)$, where $Z^1_X \subseteq \Omega^1_X$ is the subsheaf of closed forms, which holds since $H^0(X, \Omega^1_X) = 0$ by assumption;
- $H^2_{\mathrm{crys}}(X/W)[p] = 0$, which holds since $h^{0,1} = h^{1,0} = 0 = H^1_{\mathrm{dR}}(X)$ by assumption, so that the universal coefficients theorem implies $H^2_{\mathrm{crys}}(X/W)[p] = 0$;
- the F-crystals $H^2_{\mathrm{crys}}(X/W)$ is purely of slope 1, which holds since, thanks to vanishing of $\mathrm{Br}(X)$, the Picard rank of X is equal to $b_2(X) = \dim_K(H^2_{\mathrm{crys}}(X/W) \otimes \mathbb{Q})$, hence the cycle class map $\mathrm{NS}(X) \otimes K \to H^2_{\mathrm{crys}}(X) \otimes \mathbb{Q}$ is surjective.

This concludes the proof of (1).

2.3. **Proof of (2).** In this subsection, to avoid double subscripts, we will write

$$\Omega^a_X := \Omega^a(X).$$

2.3.1. *Preliminary reduction.* Let $\Omega^i_{\log}(X) \subseteq \Omega^i(X)$ be the abelian subsheaf generated by $\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(x_2) \cdots \wedge \mathrm{dlog}(x_i)$, where the x_i's are local sections of $\mathcal{O}^*_X$. By [Ill79, Proposition 3.23.2, Pag. 580], there is a natural isomorphism $\Omega^1_{\log}(X) \simeq \mathcal{O}^*_X/(\mathcal{O}^*_X)^p$, hence, by [Ill79, (5.1.4), Pag. 626], there is a natural isomorphism

$$H^2_{\mathrm{fl}}(X, \mu_p) \simeq H^1(X, \Omega^1_{\log}(X)).$$

On the other hand, by the Kummer exact sequence (2.2.2.1) and the assumption on $\mathrm{Br}(X)$, one has $\mathrm{NS}(X) \otimes \mathbb{Z}/p\mathbb{Z} \simeq H^2_{\mathrm{fl}}(X, \mu_p)$. Hence it is enough to prove that

$$\dim(H^1(X, \Omega^1_{\log}(X) \otimes k)) \geq h^2_{\mathrm{dR}}.$$

The Hodge to de-Rham spectral sequence

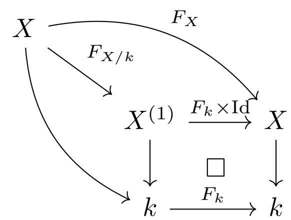
$$E^{a,b}_1 := H^b(X, \Omega^a) \Rightarrow H^2_{\mathrm{dR}}(X)$$

together with the assumption $h^{0,1} = h^{1,0} = 0$, shows that $h^{1,1} \geq h^2_{\mathrm{dR}}$, hence it is enough to show that

$$\dim(H^1(X, \Omega^1_{\log}(X)) \otimes k) = h^{1,1}. \tag{2.3.1.1}$$

The main problem in proving (2.3.1.1) is that it is unclear how to understand the cokernel of the inclusion $\Omega^1_{\log}(X) \subseteq \Omega^1(X)$ so we need to interpret in a different way $\Omega^1_{\log}(X)$ to give it a better description in terms of differential forms. This will be done in the next section, by using a result of Raynaud (Theorem 2.3.2.3) on indefinitely closed differential forms.

2.3.2. *Indefinitely closed differential forms.* In this section, we mainly follow [Ill79, 0, Sections 2.2,2.4]. Consider the commutative cartesian diagram



in which F_k, F_X are the absolute Frobenii and $F_{X/k}$ is the relative Frobenius of X/k. Set

$$Z^i(X) := \mathrm{Ker}(d : \Omega^i(X) \to \Omega^{i+1}(X)); \quad B^i(X) := \mathrm{Im}(d : \Omega^{i-1}(X) \to \Omega^i(X)); \quad \mathcal{H}^i(X) = \frac{Z^i(X)}{B^i(X)}.$$

Then one has ([Ill79, Theoreme 2.1.9, Pag. 515]) a canonical isomorphism of graded $\mathcal{O}_{X^{(1)}}$-algebras

$$C_{X/k}^{-1} : \Omega^\bullet(X^{(1)}) \simeq \mathcal{H}^\bullet(X) \tag{2.3.2.1}$$

called the Cartier isomorphism, so that there exists a canonical exact sequence

$$0 \to B^\bullet(X) \to Z^\bullet(X) \xrightarrow{C_{X/k}} \Omega^\bullet(X^{(1)}) \to 0.$$

We now define, for every integer $n \geq 0$, abelian subsheaves of $\Omega_X^\bullet$ by induction on n. For $n = 0, 1$ set

$$B_0^i(X) := 0, \quad B_1^i(X) := B^i(X), \quad Z_0^i(X) := 0, \quad Z_1^i(X) := Z^i(X),$$

and for $n > 1$ we let $B_{n+1}^i(X)$ and Z_{n+1}^i be defined by following commutative cartesian diagrams

$$\begin{array}{ccccc} B_{n+1}^i(X) & \hookrightarrow & & \longrightarrow & Z^i(X) \\ \downarrow & & \square & & \downarrow \\ B_n^i(X^{(1)}) & \hookrightarrow & \Omega^i(X^{(1)}) & \xrightarrow{C_{X/k}^{-1}} & \mathcal{H}^i(X), \end{array} \qquad \begin{array}{ccccc} Z_{n+1}^i(X) & \hookrightarrow & & \longrightarrow & Z^i(X) \\ \downarrow & & \square & & \downarrow \\ Z_n^i(X^{(1)}) & \hookrightarrow & \Omega^i(X^{(1)}) & \xrightarrow{C_{X/k}^{-1}} & \mathcal{H}^i(X), \end{array}$$

and we call $B_n^i(X) \subseteq Z_n^i(X)$ the sheaf of i-form that are n-exact and n-closed respectively. By construction (see [Ill79, Pag. 519-520]) there is a chain of inclusions

$$0 \subseteq B_1^i(X) \subseteq B_2^i(X) \subseteq \cdots \subseteq B_n^i(X) \subseteq \cdots \subseteq Z_{n+1}^i(X) \subseteq Z_n^i(X) \subseteq \cdots \subseteq Z_2^i(X) \subseteq Z_1^i(X) \subseteq \Omega^i(X)$$

Finally, set

$$\bigcup_{n \in \mathbb{N}} B_n^i := B_\infty^i(X) \subseteq \bigcap_{n \in \mathbb{N}} Z_n^i(X) := Z_\infty^i(X)$$

which we call the sheaves of i-form that are indefinitely-exact and indefinitely-closed respectively.

Since k is perfect, there is a Cartier morphism $C_X : Z^i(X) \to \Omega^i(X)$ (see e.g. [Ill79, (2.1.21), Pag. 518] which, by [Ill79, Section 2.5.1, Pag. 531], sends $Z_\infty^i(X)$ to itself, hence it induces an endomorphism of C_X of $Z_\infty^i(X)$. By [Ill79, (2.5.3.3) Pag. 532], there is a canonical decomposition

$$Z_\infty^i(X) = B_\infty^i(X) \oplus (Z_\infty^i(X))_{ss}, \tag{2.3.2.2}$$

where $(Z_\infty^i(X))_{ss}$ is characterized by the fact that for every open affine subset $U \subseteq X$ the vector space $(Z_\infty^i(X))_{ss}(U)$ identifies with union of the finite-dimensional subspaces of $Z_\infty^i(U)$ stable under C and on which C is an automorphism.

With this notation, we can recall from [Ill79, (2.5.3.5) Pag. 533] the main result we will use.

Theorem 2.3.2.3 (Raynaud)**.** *There is a natural isomorphism of sheaves*

$$\Omega_{\log}^i(X) \otimes k \simeq (Z_\infty^i(X))_{ss}.$$

2.3.3. *Proof of (2.3.1.1).* We can now prove (2.3.1.1). Thanks to Theorem 2.3.2.3 it is enough to prove that

$$\dim_k(H^1(X, Z^1_\infty(X))_{ss})) = h^{1,1}. \tag{2.3.3.1}$$

To conclude to proof, we show that there is a chain of isomorphisms:

$$H^1(X, \Omega^1(X)) \overset{(i)}{\simeq} H^1(X, Z^1(X)) \overset{(ii)}{\simeq} H^1(X, Z_\infty(X)) \overset{(iii)}{\simeq} H^1(X, Z^1_\infty(X)_{ss}).$$

We prove first (i), then (iii) and then (ii).

(i) By taking the long exact sequence associated with the short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{(-)^p} \mathcal{O}_X \xrightarrow{d} B^1(X) \to 0,$$

one sees that the assumptions on X imply that

$$H^i(X, B^1(X)) = 0 \quad \text{for} \quad 0 \leq i \leq 2. \tag{2.3.3.2}$$

Then the exact sequence

$$0 \to B^1(X) \to Z^1(X) \xrightarrow{C} \Omega^1(X^{(1)}) \to 0$$

shows that

$$H^1(X, Z^1(X)) \simeq H^1(X^{(1)}, \Omega^1(X^{(1)})) \simeq H^1(X, \Omega^1(X))),$$

where the last equality follows from the perfectness of k.

(iii) Thanks to the decomposition (2.3.2.2), one has

$$H^1(X, Z^1_\infty(X)) = H^1(X, B^1_\infty(X)) \oplus H^1(X, (Z^i_\infty(X))_{ss}),$$

so that it is enough to show that

$$H^1(X, B^1_\infty(X)) = 0.$$

Since

$$B^1_\infty(X) = \bigcup_{n\in\mathbb{N}} B^1_n(X) = \varinjlim_n B^1_n(X)$$

one has

$$H^1(X, B^1_\infty(X)) = \varinjlim_n H^1(X, B^1_n(X)),$$

hence it is enough to show that $H^i(X, B^1_n(X)) = 0$ for every n and $0 \leq i \leq 2$. We prove this by induction on n, the case $n = 1$ being (2.3.3.2). So assuming that $n > 1$ and $H^i(X, B^1_m(X)) = 0$ for every $m \leq n$ and $0 \leq i \leq 2$ we have to show that $H^i(X, B^1_{n+1}(X)) = 0$. By [Ill79, End of page 531], one has a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^2_1(X) & \longrightarrow & B^2_n(X) & \xrightarrow{C^{-1}} & B^2_{n-1}(X) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^2_1(X) & \longrightarrow & B^2_{n+1}(X) & \xrightarrow{C^{-1}} & B^2_n(X) & \longrightarrow & 0, \end{array} \tag{2.3.3.3}$$

which, taking the associated long exact sequence, shows the induction step.

(ii) Since

$$Z^1_\infty(X) = \bigcap_{n\in\mathbb{N}} Z^1_n(X) = \varprojlim_n Z^1_n(X),$$

by [Jan88, Proposition 1.6] there is a canonical exact sequence

$$0 \to \varprojlim_n{}^1(H^0(X, Z^1_n(X))) \to H^1(X, Z^1_\infty(X)) \to \varprojlim_n H^1(X, Z^1_n(X)) \to 0.$$

Since $H^0(X, Z^1_n(X))) \subseteq H^0(X, \Omega^1(X))) = 0$ by assumption, it is enough to prove that the map $H^1(X, Z^1_{n+1}(X)) \to H^1(X, Z^1_n(X))$ induced by the inclusion $Z^1_n(X) \subseteq$

$Z^1_{n-1}(X)$ is an isomorphism. To prove this, we first remark that, by [Ill79, Pag. 531] there is an exact sequence

$$0 \to B^1_n(X) \to Z^1_{n+1}(X) \xrightarrow{C} Z^1_n(X) \to 0,$$

hence, by (2.3.3.2), one gets $\dim(H^1(X, Z_{n+1}(X))) = \dim(H^1(X, Z_n(X)))$, so that we are reduced to show that $H^1(X, Z^1_{n+1}(X)) \to H^1(X, Z^1_n(X))$ induced by the inclusion $Z^1_{n+1}(X) \subseteq Z^1_n(X)$ is injective. By [Ill79, (2.2.6.2), Pag 520], the inclusion $Z^1_{n+1}(X) \subseteq Z^1_n(X)$ sits in an exact sequence

$$0 \to Z^1_{n+1}(X) \to Z^1_n(X) \to B^2_{n+1}(X)/B^2_n(X) \to 0,$$

so that it is enough to show that $H^0(B^2_{n+1}(X)/B^2_n(X)) = 0$. The diagram (2.3.3.3) shows that

$$B^2_{n+1}(X)/B^2_n(X) \simeq B^2_n(X)/B^2_{n-1}(X) \simeq B^2_1(X)/B^2_0(X) \simeq B^2(X).$$

Hence

$$H^0(X, B^2_{n+1}(X)/B^2_n(X)) \simeq H^0(X, B^2(X)) \subseteq H^0(X, \Omega^2(X)) = 0.$$

This concludes the proof of (2) and hence of Theorem 1.3.1.

3. A STABLY IRRATIONAL VARIETY REDUCING TO A RATIONAL VARIETY

Let R be the ring of integers of $K := \mathbb{C}_p$ and k its residue field. In this last section, we show how to construct, for every $p \gg 0$, examples of smooth projective schemes X/R such that X_K is not stably-rational and such that X_k is rational, as it was suggested to us by Colliot-Thélène. The construction uses and it based on the analogous construction in [HPT18] of a family of smooth proper varieties over the complex number with stably irrational general fiber but with some rational fiber.

3.1. **A general lemma.** We begin with a general lemma, which reduces the construction of examples to the construction of mixed characteristic families with easier-to-check properties. Let B/R be smooth with geometrically integral fibres and $X \to B$ a smooth projective family of varieties.

Lemma 3.1.1. *Assume that there exists a point $b \in B(\mathbb{C}_p)$ such that X_b is not stably-rational. Then, for every $a \in B(k)$ there exists a lift $b' \in B(R)$ of a such that $X_{b'}$ is not stably rational.*

Proof. By [NO21, Corollary 4.1.2], the set

$$B(\mathbb{C}_p)_r := \{b \in B(\mathbb{C}_p) \colon X_b \text{ is stably-rational}\}$$

is a countable union of closed subvarieties. Define now $B(\mathbb{C}_p)_{nr} := B(\mathbb{C}_p) \setminus B(\mathbb{C}_p)_r$. By the assumption on b, the set $B(\mathbb{C}_p)_r$ is the countable union of *proper* closed subvarieties.

Since, by Hensel lemma, the map $\pi : B(R) \to B(k)$ is surjective we can choose a lift b'' of a. The set $\pi^{-1}(a) \subseteq B(R) \subseteq B(K)$ is an open neighborhood of b'' in $B(K)$. Since $B(\mathbb{C}_p)_r$ is the countable union of proper closed subvarieties, we can apply [MP12, Lemma 4.29] to deduce that there exists a $b' \in B(\mathbb{C}_p)_{nr} \cap \pi^{-1}(a)$. This concludes the proof. □

3.2. **An example.** By Lemma 3.1.1, to construct a smooth projective scheme X/R such that X_K is not stably-rational and such that X_k is rational, it is enough to construct a family $X \to B$ over R such that there exist points $b \in B(\mathbb{C}_p)$ and $a \in B(k)$ such that X_b is stably irrational and X_a is rational. Such a family can be constructed using directly Hassett, Pirutka, and Tschinkel example [HPT18], see also [CTS21, Section 12.2.2] and [Sch19]. We give some details.

Let $X' \to Z$ be the universal family of quadric bundles over $\mathbb{P}^2_{\mathbb{Q}}$ given in $\mathbb{P}^3_{\mathbb{Q}} \times \mathbb{P}^2_{\mathbb{Q}}$ by a bihomogeneous form of bidegree $(2,2)$. After choosing coordinates x, y, z and U, V, W, T on $\mathbb{P}^3_{\mathbb{Q}} \times \mathbb{P}^2_{\mathbb{Q}}$, the variety Z identifies with the space of bihomogeneous forms $F = F(x, y, z, U, V, W, T)$ of bidegree $(2,2)$ in $\mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^3_{\mathbb{Q}}$ which are symmetric quadratic forms in the variables U, V, W, T, since any such F determines a quadratic bundle over $\mathbb{P}^2_{\mathbb{Q}}$ via the projection $\mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^3_{\mathbb{Q}} \to \mathbb{P}^2_{\mathbb{Q}}$. In turn, these forms are given by a 4×4 symmetric matrix $A = (a_{i,j})_{1 \leq i,j \leq 4}$ where each entry $a_{i,j} = a_{i,j}(x, y, z)$ is a homogeneous polynomial of degree 2.

By the arguments in [Sch19] and Bertini theorem, there exists a dense open Zariski $B_{\mathbb{Q}} \subset Z$ such that the restriction of the family $X_{\mathbb{Q}} \to B_{\mathbb{Q}}$ to $B_{\mathbb{Q}}$ parametrizes conic bundle flat over $\mathbb{P}^2_{\mathbb{Q}}$ and with smooth total space.

By spreading out, this construction extends to give a smooth family $X \to B$ over $\mathbb{Z}[1/n]$ for n big enough whose base change to $\mathbb{Q}$ identifies with $X_{\mathbb{Q}} \to B_{\mathbb{Q}}$. By the main result of [HPT18], the general fiber of $X(\mathbb{C}) \to B(\mathbb{C})$ is not stably rational, hence, for every p, there exists a $b \in B(\mathbb{C}_p)$ such that Y_b is not stably rational. So, we are left to show that for $p \gg 0$, there exists a $a \in B(\overline{\mathbb{F}}_p)$ such that X_a is rational.

Using Bertini, there exists of a rational point $r \in B(\mathbb{Q})$ such that the corresponding 4×4 symmetric matrix $A = (a_{i,j})_{1 \leq i,j \leq 4}$ has $a_{1,1} = 0$. By spreading out we can choose $p \gg 0$ such that $r \in B(\mathbb{Q})$ extends to a point $\widetilde{a} \in B(\mathbb{Z}_p)$ whose reduction a modulo p defines a flat conic bundle $X_a \to \mathbb{P}^2_{\mathbb{F}_p}$ with smooth total space, whose associated matrix has $a_{1,1} = 0$. Since $a_{1,1} = 0$, the morphism $X_a \to \mathbb{P}^2_{\mathbb{F}_p}$ has a rational section, hence X_a is rational. This concludes the construction of a smooth proper scheme over R with rational special fiber and stably irrational generic fiber.

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