WILD BRAUER CLASSES VIA PRISMATIC COHOMOLOGY

EMILIANO AMBROSI, RACHEL NEWTON, AND MARGHERITA PAGANO

Abstract. Let K be a finite extension of $\mathbb{Q}_p$ and X a smooth proper K-variety with good reduction. Under a mild assumption on the behaviour of Hodge numbers under reduction modulo p, we prove that the existence of a non-zero global 2-form on X implies the existence of p-torsion Brauer classes with surjective evaluation map, after a finite extension of K. This implies that any smooth proper variety over a number field which satisfies weak approximation over all finite extensions has no non-zero global 2-form. The proof is based on a prismatic interpretation of Brauer classes with eventually constant evaluation, and a Newton-above-Hodge result for the mod p reduction of prismatic cohomology. This generalises work of Bright and the second-named author beyond the ordinary reduction case.

Contents

1. Introduction

Let K be a finite extension of $\mathbb{Q}_p$, with residue field k. Let X be a smooth proper variety over K with good reduction. In this paper we use recent developments in p-adic cohomology to show the existence (and in some cases construction) of p-torsion Brauer classes of arithmetic interest.

1.1. **Evaluation map.** To be more precise, recall that each $\mathcal{A} \in \mathrm{Br}(X)$ induces, for every finite field extension L/K, an evaluation map

$$\mathrm{ev}_{\mathcal{A}}^{L} : X(L) \to \mathrm{Br}(L) \simeq \mathbb{Q}/\mathbb{Z}$$

sending $Q \in X(L)$ to $Q^*\mathcal{A}$. Classes in the Brauer group of a variety over a number field M with non-constant evaluation on the points over a completion of M obstruct weak approximation for M-rational points (see e.g. [CS21, Chapter 13]). Hence, it is of arithmetic interest to construct or show the existence of such classes. Brauer classes whose evaluation maps remain non-constant over all finite extensions of K are necessarily of transcendental nature, meaning that they do not vanish in $\mathrm{Br}(X_{\bar{K}})$.

Date: September 26, 2025.

In general, the construction of transcendental Brauer classes of arithmetic interest is a hard problem. Over number fields, the first example of such a transcendental element was given by Wittenberg in [Wit04]. Since then, only a handful of explicit examples of transcendental elements obstructing weak approximation have been constructed, see e.g. [HVV11; Ier10; Pre13; New16; New24; BV20; IS15; IS17; AN25].

It was only in [BN23, Theorem C] that a first general existence result was shown. There, Bright and the second-named author show that, upon replacing K with a finite field extension, transcendental classes with non-constant evaluation map exist as soon as the reduction of the variety has a non-zero global 2-form and it is *ordinary* in the sense of Bloch–Kato [BK86]. In [Pag25], the third-named author gives examples showing how one can construct an element in the Brauer group obstructing weak approximation starting from a non-zero global 2-form on the special fibre. These include some examples with non-ordinary reduction, indicating that the ordinary condition in [BN23, Theorem C] is not necessary.

1.2. **Main result.** The goal of this paper is to extend the aforementioned result of Bright–Newton, replacing the ordinary condition with the folllowing very mild condition on the Hodge numbers of the special and generic fibres:

$$\heartsuit \qquad \dim_k(\mathrm{H}^i(Y, \Omega^j_{Y/k})) = \dim_K(\mathrm{H}^i(X, \Omega^j_{X/K})) \quad \text{for every } i, j \geq 0,$$

where Y/k is the special fibre with respect to a smooth proper model of X.

Theorem 1.2.1. *Let $\mathcal{X}$ be a smooth proper model of X with special fibre Y. Assume that $\heartsuit$ holds. If $\mathrm{H}^0(X, \Omega^2) \neq 0$, then there exists a finite field extension L/K and $\mathcal{A} \in \mathrm{Br}(X_L)[p]$ such that $\mathrm{ev}^F_{\mathcal{A}} : X(F) \to \mathrm{Br}(F)[p]$ is surjective for every finite field extension F/L.*

Since classes $\mathcal{A} \in \mathrm{Br}(X)$ with prime-to-p order have constant evaluation map after a finite extension, Theorem 1.2.1 gives classes of the smallest possible order.

If Z is a variety over a number field such that $\mathrm{H}^0(Z, \Omega^2) \neq 0$, then, by generic freeness, the base change of Z to the completion at a place satisfies $\heartsuit$ for all but finitely many places. As a consequence, in Corollary 1.2.2 below, we answer a question of Wittenberg (see [BN23, Question 1.4]), which is a special case of his more general question asking whether varieties over number fields that satisfy the Hasse principle and weak approximation over all finite extensions are geometrically rationally connected, hence have vanishing extremal Hodge numbers.

Corollary 1.2.2. *Let Z be a smooth proper variety over a number field satisfying weak approximation over all finite extensions. Then $\mathrm{H}^0(Z, \Omega^2) = 0$.*

Theorem 1.2.1 actually implies something stronger. To state it, let Z be a variety over a number field M. We say that a place v of M is potentially relevant to the Brauer–Manin obstruction to weak approximation on Z if there exist a finite extension N/M, a place v' of N lying over v, and $\mathcal{A} \in \mathrm{Br}(Z_N)$ such that $\mathrm{ev}^{N_{v'}}_{\mathcal{A}}$ is non-constant. Thus, $\mathcal{A}$ obstructs weak approximation over N as soon as Z has an N-adelic point. Thanks to the invariance of étale cohomology under algebraically closed field extensions, Theorem 1.2.1 implies the following:

Corollary 1.2.3. *Let Z be a smooth proper variety over a number field M such that $\mathrm{H}^0(Z, \Omega^2) \neq 0$. Then all but finitely many places of M are potentially relevant to the Brauer–Manin obstruction to weak approximation on Z.*

For some important class of varieties, such as abelian varieties, K3 surfaces, complete intersections in products of projective spaces, assumption ♥ is satisfied at all places of good reduction for Z. By way of contrast with the ordinary reduction hypothesis in [BN23, Theorem C], recall that if Z is an elliptic curve over $\mathbb{Q}$, then, by [Elk87], there are infinitely many places of supersingular reduction (of positive density if Z has complex multiplication), while for abelian varieties of dimension ≥ 4, the existence of primes of good ordinary reduction is still an open question.

The methods used in the proof of Theorem 1.2.1 allow us to go beyond merely showing the existence of arithmetically interesting Brauer classes. For products of elliptic curves, we show how to construct such classes, generalising previous constructions in the literature, see Proposition 1.5.1 and Corollary 6.2.3. For abelian varieties of positive p-rank and their associated Kummer varieties, we give lower bounds for the number of interesting Brauer classes, see Theorem 1.5.3. In particular, our results show that there are often more interesting classes in the non-ordinary reduction cases than in the ordinary one. See Section 1.5 for more details.

1.3. **Strategy.**

1.3.1. *Cohomological interpretation.* The starting point of the proof of Theorem 1.2.1 is a cohomological interpretation of Brauer classes with non-constant evaluation map, which can be deduced by pushing the techniques in [BN23]. Consider the following subgroup of $\mathrm{Br}(X_{\overline{K}})[p]$ of "geometrically boring" classes

$$\mathrm{Br}(X_{\overline{K}})^{gb} := \bigcup_{L/K \text{ finite}} \left\{ \mathcal{A}_{\overline{K}} \mid \mathcal{A} \in \mathrm{Br}\, X_L \text{ and } \exists F/L \text{ finite such that } \mathrm{ev}_{\mathcal{A}}^F \text{ is constant} \right\}.$$

The conclusion of Theorem 1.2.1 is equivalent to the statement that $\mathrm{Br}(X_{\overline{K}})^{gb}[p] \neq \mathrm{Br}(X_{\overline{K}})[p]$. Writing $\overline{K}(X)^{\mathrm{sh}}$ for the strict henselisation of the function field of $X_{\overline{K}}$, with respect to the p-adic valuation, one proves the following:

Theorem 1.3.1. *Assume that X has good reduction. Then there is a natural isomorphism*

$$\mathrm{Br}(X_{\overline{K}})[p]/\,\mathrm{Br}(X_{\overline{K}})^{gb}[p] = \mathrm{Im}(\mathrm{H}^2(X_{\overline{K}}, \mathbb{Z}/p) \to \mathrm{H}^2(\overline{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p)).$$

1.3.2. *Prismatic interpretation.* With Theorem 1.3.1 in hand, we can apply the machinery of prismatic cohomology. Let $\mathcal{O}_{\mathbb{C}_p^\flat}$ be the tilt of the ring of integers $\mathcal{O}_{\mathbb{C}_p}$ of $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$, which is (non-canonically) isomorphic to the completion of the algebraic closure of the power series ring in one variable over k. Our arguments will concern $\mathrm{H}^*(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$, the reduction modulo p of the prismatic cohomology theory defined in [BS22]. Under assumption ♥, it is a finite free $\mathcal{O}_{\mathbb{C}_p^\flat}$-module (Proposition 3.2.2), endowed with a Frobenius $\varphi : \mathrm{H}^*(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \to \mathrm{H}^*(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$. Let $d \in \mathcal{O}_{\mathbb{C}_p^\flat}$ be the element defined in (3.1.1). Building on the work in [BMS18; BMS19; BS22] and, in particular, using the syntomic interpretation of p-adic vanishing cycles given in [BMS19, Section 10], we prove the following:

Theorem 1.3.2. *Let $\mathfrak{X}$ be a smooth proper model of X with special fibre Y. Assume that ♥ holds. Then one has*

$$\mathrm{Ker}(\mathrm{H}^n(X_{\overline{K}}, \mathbb{Z}/p) \to \mathrm{H}^n(\overline{K}(X)^h, \mathbb{Z}/p)) \simeq \mathrm{Ker}(\varphi - d^{n-1} : \mathrm{H}^n(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \to \mathrm{H}^n(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p))$$

Thanks to the étale comparison theorem for prismatic cohomology and to Theorems 1.3.1 and 1.3.2, Theorem 1.2.1 is implied by the following:

Theorem 1.3.3. *Let $\mathcal{X}$ be a smooth proper model of X with special fibre Y. Assume that ♥ holds. If* $\mathrm{H}^i(X,\Omega^{2n-i}) \neq 0$ *for some $i \neq n$, then*

$$dim_{\mathbb{F}_p}(\mathrm{Ker}(\varphi - d^n : \mathrm{H}^{2n}(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \to \mathrm{H}^{2n}(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p))) < rank_{\mathcal{O}_{\mathbb{C}_p^\flat}}(\mathrm{H}^{2n}(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)).$$

Remark 1.3.4. Although Theorems 1.3.2 and 1.3.3 are true for an arbitrary value of n, the only cohomology group for which they can be used together is $\mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$. See also Remark 1.4.2.

1.4. **Strategy for the proof of Theorem 1.3.3.** To simplify the discussion, we assume in this subsection that $n = 1$. One can show that if $\mathrm{H}^0(X, \Omega^2) \neq 0$ then $\varphi : \mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \to \mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$ is not zero when tensored with $\overline{k}$. However, this information is not enough to control the dimension of $\mathrm{Ker}(\varphi - d)$, as Example 5.1.2 shows. Hence, a more refined control on the action of the Frobenius is needed and we achieve this via a Newton-above-Hodge type of argument.

Choose a compatible system $\{d^a\}_{a\in\mathbb{Q}}$ of roots of d in $\mathcal{O}_{\mathbb{C}_p^\flat}$. For every positive rational number $a \in \mathbb{Q}$, we let $\mathcal{O}(-a)$ be the free $\mathcal{O}_{\mathbb{C}_p^\flat}$-module of rank 1, with generator e, on which there is a semi-linear Frobenius acting as $\varphi(e) = d^a e$. First, we show in Proposition 4.1.3 that $\mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$ admits a Frobenius-equivariant decreasing filtration F^j such that $F^j/F^{j+1} \simeq \mathcal{O}(-a_j)$ for some $a_j \in \mathbb{Q}$. While these a_j might depend on the choice of the filtration (see Example 4.1.7), we show in Proposition 4.1.3(2) that their sum does not. Hence, we define the total slope $\mathrm{TS}(\mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p))$ of $\mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$ to be this sum. Some (semi-)linear algebra over $\mathcal{O}_{\mathbb{C}_p^\flat}$ (Lemma 5.1.1) reduces the proof of Theorem 1.3.3 to showing the equality

$$\mathrm{TS}(\mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)) = \mathrm{rank}(\mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)).$$

The following theorem, whose proof is inspired by [Yao23], proves a more general result, which should be thought of as a "Newton-above-Hodge" statement and may be of independent interest.

Theorem 1.4.1. *Let $\mathcal{X}$ be a smooth proper model of X with special fibre Y. Assume that ♥ holds. Then*

$$\mathrm{TS}(\mathrm{H}^n(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)) = \sum_{i=0}^{n} i \cdot h^{i,n-i}.$$

Remark 1.4.2. This is a continuation of Remark 1.3.4. If one wants to study $\mathrm{Ker}(\varphi - d^{n-1} : \mathrm{H}^n(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \to \mathrm{H}^n(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p))$ for an arbitrary value of n and prove that it has small dimension, Theorem 1.4.1 is not enough, since the sum of Hodge numbers appearing there is too small to use the arguments in Lemma 5.1.1. It seems likely that to get that the dimension of $\mathrm{Ker}(\varphi - d^{n-1} : \mathrm{H}^n(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \to \mathrm{H}^n(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p))$ is small for bigger n, one needs extra conditions on p, as shown in Example 5.2.1. This seems to be compatible with the results in [FKW24].

1.5. **Special cases.** We make Theorem 1.2.1 more explicit in certain families of abelian varieties and their associated Kummer varieties. For abelian varieties, one can use prismatic Dieudonné theory, from [AL23], to make prismatic cohomology more concrete.

1.5.1. *Products of elliptic curves.* By combining Theorems 1.3.1 and 1.3.2 with prismatic Dieudonné theory, we prove the following:

Proposition 1.5.1. *Let $X = Z\times W$ for elliptic curves Z, W with good reduction and Néron models $\mathcal{Z},\mathcal{W}$. Then there exists a natural isomorphism*

$$\mathrm{Br}(X_{\overline{K}})[p]/\,\mathrm{Br}(X_{\overline{K}})^{gb}[p] \simeq \mathrm{Hom}_{\mathbb{C}_p}(Z[p], W[p])/\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p], \mathcal{W}[p]).$$

Remark 1.5.2. Proposition 1.5.1 is a special case of a more general result on products of varieties $X = Z \times W$, which states that, under assumption ♥, one has an injection

$$\frac{\mathrm{Hom}_{\mathbb{C}_p}(\mathrm{Pic}^0(Z)[p], \mathrm{Alb}(W)[p])}{\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathrm{Pic}^0(\mathcal{Z})[p], \mathrm{Alb}(\mathcal{W})[p])} \subseteq \frac{\mathrm{Br}(X_{\overline{K}})[p]}{\mathrm{Br}(X_{\overline{K}})^{gb}[p]},$$

where $\mathrm{Alb}(W)$ is the Albanese variety and $\mathrm{Alb}(\mathcal{W})$ is its Néron model. The proof of this is a bit more subtle, since it requires the study of the behaviour of the reduction of the Albanese variety, which in turn requires an explanation of how to use the results in [LL23]. For the sake of simplicity, we restrict ourselves to the case of elliptic curves, where the proof is easier.

Abusing notation, we write $\mathrm{Hom}_{\overline{K}}(Z, W)$ for its image in $\mathrm{Hom}_{\overline{K}}(Z[p], W[p])$. Recalling that $\mathrm{Br}(X_{\overline{K}})[p] \simeq \mathrm{Hom}_{\overline{K}}(Z[p], W[p])/\mathrm{Hom}_{\overline{K}}(Z, W)$, Proposition 1.5.1 shows that an element in $\mathrm{Br}(X_{\overline{K}})[p]$ lies in $\mathrm{Br}(X_{\overline{K}})^{gb}[p]$ if and only if the corresponding element in $\mathrm{Hom}_{\overline{K}}(Z[p], W[p])$ lifts to the integral model over $\mathcal{O}_{\mathbb{C}_p}$. Since X has trivial canonical bundle, combining this with Proposition 2.2.3, we get that a Brauer class in $\mathrm{Br}(X)[p]$ associated to a homomorphism $\sigma : Z[p] \to W[p]$ defined over K has non-constant evaluation map (over K) as soon as σ does not lift to the integral models over $\mathcal{O}_{\mathbb{C}_p}$.

This gives a criterion for having non-constant evaluation map that can be applied to concrete Brauer classes. For example, when $Z = W$ is a CM elliptic curve, we study the Brauer class associated to the action of complex conjugation and show how to reinterpret and generalise examples constructed in [New16; IS15; AN25]. In this setting, we show in Corollary 6.2.6 that in most cases of supersingular reduction one has $\mathrm{Br}(X_{\overline{K}})^{gb}[p] = 0$, which cannot happen in the ordinary reduction case (cf. Corollary 6.2.3).

1.5.2. *Abelian varieties of positive p-rank and associated Kummer varieties.* Now suppose that X is an abelian variety. Since $\mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \simeq \Lambda^2\mathrm{H}^1(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$, prismatic Dieudonné theory can be used once again to get more explicit results. We prove a lower bound for the quotient $\mathrm{Br}(X_{\overline{K}})[p]/\,\mathrm{Br}(X_{\overline{K}})^{gb}[p]$. Moreover, we show that this bound can be transferred to the Kummer variety associated to an $X[2]$-torsor T over k.

Theorem 1.5.3. *Let X be an abelian variety of dimension $g \geq 2$ with good reduction and assume that the special fibre has p-rank $e > 0$. Then*

$$dim_{\mathbb{F}_p}(\mathrm{Br}(X_{\overline{K}})[p]/\,\mathrm{Br}(X_{\overline{K}})^{gb}[p]) \geq 2g - 1 - e.$$

Furthermore, if p is odd, then for any $X[2]$-torsor T over k,

$$dim_{\mathbb{F}_p}(\mathrm{Br}(\mathrm{Kum}(X_T)_{\overline{K}})[p]/\,\mathrm{Br}(\mathrm{Kum}(X_T)_{\overline{K}})^{gb}[p]) \geq 2g - 1 - e.$$

where $\mathrm{Kum}(X_T)$ is the Kummer variety associated to T.

The bounds given in Theorem 1.5.3 are attained when X is a product of two elliptic curves, see Corollary 6.2.3.

Remark 1.5.4. In Theorem 1.2.1, under assumption ♥, the hypothesis on the existence of global differential 2-forms is necessary. Indeed, if $\mathrm{H}^0(Y, \Omega^2) = 0$, then one has $\mathrm{Br}(X_{\overline{K}})[p] = \mathrm{Br}(X_{\overline{K}})^{gb}[p]$, as follows by combining Corollary 2.3.1 with [BK86, Theorem 8.1 and (8.0.1)]. On the other hand, we do not know whether the assumption ♥ is really necessary. The first interesting test case is a non-classical Enriques surface over a finite extension of $\mathbb{Q}_2$ with good reduction, for which, to the best of our knowledge, not many results exist in the literature. In this situation, torsion in étale cohomology has to be treated with different techniques. We hope to return to this problem in the near future.

1.6. **Organisation of the paper.** In Section 2 we study the relationship between Brauer classes with non-constant evaluation, henselisation and the p-adic vanishing cycles. In Section 3, after recalling some preliminaries on prismatic cohomology, we relate p-adic vanishing cycles (hence Brauer classes with non-constant evaluation) with prismatic cohomology. In Section 4, we prove a Newton-versus-Hodge type of theorem for modulo p prismatic cohomology. In Section 5 we put everything together to prove the main Theorem 1.2.1. Finally, in Section 6 we use the previous techniques to study the case of products of elliptic curves and abelian varieties, where the results are more explicit.

1.7. **Acknowledgements.** This research was partly supported by the grant ANR–23–CE40–0011 of Agence Nationale de la Recherche. Rachel Newton was supported by UKRI Future Leaders Fellowship MR/T041609/1 and MR/T041609/2. Part of this work was done during the special trimester *Arithmetic geometry of K3 surfaces* at the Bernoulli Center for Fundamental Studies. We thank the organisers, as well as Stefan Schröer for a very helpful discussion on a filtration on the p-torsion of elliptic curves, and Livia Grammatica for a very nice talk on the Nygaard filtration in positive characteristic. We also thank Arthur-César Le Bras and Matthew Morrow for useful email exchanges.

1.8. **Conventions.** If L is a valuation field, we write $\mathcal{O}_L$ for the valuation ring and $\mathfrak{m}_L \subseteq \mathcal{O}_L$ for the maximal ideal and k_L for the residue field. If $\mathcal{O}_L$ is a discrete valuation ring, we write π_L for an uniformiser and we omit the subscript $_L$ if it is clear from the context. If K is a p-adic field, we write $\mathbb{C}_p$ for the completion of an algebraic closure of K.

If $R \to S$ is a morphism of rings and X is a scheme over R, we write $X_S := X \times_R S$. If X is a scheme and A a ring, we write $\mathrm{Sh}(X, A)$ for the category of étale sheaf on X with coefficient in A and $D^b(X, A)$ for its bounded derived category. We omit A if $A = \mathbb{Z}$. If $\mathcal{F}^\bullet$ is in $D^b(X, A)$ and $n \in \mathbb{Z}$, we write $\tau_{\leq n}\mathcal{F}^\bullet$ and $\tau_{>n}\mathcal{F}^\bullet$ for the canonical truncation, so that there is an exact triangle

$$\tau_{\leq n}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \tau_{>n}\mathcal{F}^\bullet.$$

We let $\mathcal{H}^i(\mathcal{F}^\bullet)$ be the i^{th} cohomology sheaf of $\mathcal{F}^\bullet$ and $\mathrm{H}^i(X, \mathcal{F}^\bullet)$ be the hypercohomology of $\mathcal{F}^\bullet$. We omit X if it is clear from the context. We write $\mathcal{F}^\bullet/p := \mathcal{F}^\bullet \otimes^L \mathbb{Z}/p$ for the derived tensor product with $\mathbb{Z}/p$. This gives an exact functor

$$(-) \otimes^L \mathbb{Z}/p : D^b(X, A) \to D^b(X, A/p).$$

Most of the time, when talking about global differential forms on varieties, we will suppress from the notation the subscript keeping track of the variety itself. This will not be the case in Section 3, where we work with differential forms and de Rham complex on schemes over a given base ring.

2. Brauer classes with non-constant valuation map

In this section, we prove Theorem 1.3.1 and its Corollary 2.3.1 that gives a cohomological interpretation of geometrically boring Brauer classes. In order to do this, we start in Section 2.1 by recalling some results and notation from [BN23] about refined Swan conductors. In Section 2.2, we use this theory to prove Theorem 1.3.1, from which we deduce Corollary 2.3.1.

2.1. **Filtration on the Brauer group and refined swan conductor.** Let K be a finite extension of $\mathbb{Q}_p$ with residue field k. Fix a uniformiser π of $\mathcal{O}_K$. Let $\mathcal{X}/\mathcal{O}_K$ be a smooth proper scheme with special fibre Y/k and generic fibre X/K. Let $K(X)$ be function field of X and let $K(X)^{\mathrm{h}}$ its henselisation with respect to the p-adic place and $K(X)^{\mathrm{sh}}$ its strict henselisation. Let $g_K : \mathrm{Br}(X)[p] \to \mathrm{Br}(K(X)^{\mathrm{sh}})[p]$ be the restriction map.

From [BN23], we learn that by pulling back Kato's [Kat89] Swan conductor filtration defined on $\mathrm{H}^2(K(X)^h, \mathbb{Z}/p(1)) \simeq \mathrm{Br}(K(X)^h)[p]$ to $\mathrm{Br}(X)[p]$, we get a finite exhaustive increasing filtration

$$0 \subseteq \mathrm{fil}_0(X) \subseteq \ldots \mathrm{fil}_i(X) = \mathrm{Br}(X)[p].$$

This filtration is such that

$$\mathrm{fil}_0(X) = \mathrm{Ker}(\mathrm{Br}(X)[p] \to \mathrm{Br}(K(X)^{\mathrm{sh}})[p]) \tag{2.1.1}$$

and there are maps, realised as pullbacks of the analogous maps defined by Kato [Kat89] at the level of the henselian field $K(X)^h$,

$$\partial : \mathrm{fil}_0(X) \to \mathrm{H}^1(Y, \mathbb{Z}/p) \ \text{ and } \ \mathrm{rsw}_{i,\pi} : \mathrm{fil}_i(X) \to \mathrm{H}^0(Y, \Omega^1) \oplus \mathrm{H}^0(Y, \Omega^2), \ \text{for } i \geq 1,$$

such that $\mathrm{fil}_i = \ker(\mathrm{rsw}_{i+1,\pi})$. In [BN23] Bright and Newton show that these maps control arithmetic properties of the elements in the Brauer group. In particular, they satisfy the following properties.

Theorem 2.1.2. *[BN23, Corollary 3.7 and Theorem B]*

(1) The kernel of the residue map ∂ coincides with $\mathrm{Br}(\mathcal{X})[p]$;
(2) Let $n \geq 1$ and $\mathcal{A} \in \mathrm{fil}_n(X)$ with $\mathrm{rsw}_{n,\pi}(\mathcal{A}) = (\alpha, \beta) \in \mathrm{H}^0(Y, \Omega^2) \oplus \mathrm{H}^0(Y, \Omega^1)$. If there exists $P_0 \in Y(k)$ such that $(\alpha_{P_0}, \beta_{P_0}) \neq 0$, then $\mathrm{ev}_{\mathcal{A}} : X(K) \to \mathrm{Br}(K)[p]$ is surjective.

In order to prove Theorem 1.3.1, we need to give more details on the construction of the residue map ∂ and the refined Swan conductor maps $\mathrm{rsw}_{n,\pi}$ at the level of the henselian field $K(X)^h$.

2.1.1. *The maps λ_π and δ.* Let M^h be a henselian discrete valuation field containing $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_{M^h}$ and residue field F of characteristic $p > 0$. Fix a uniformiser π and, to simplify the discussion, assume that π is algebraic over $\mathbb{Q}_p$.

Following [Kat89], we write

$$\mathbb{Z}/p(q) := \Omega^q_{\log}[-q] \text{ in } D^b(F) \quad \text{and} \quad \mathbb{Z}/p(q) := \mathbb{Z}/p(1)^{\otimes q} \text{ in } D^b(M^h)$$

for the (q-shifted) sheaf of q-logarithmic differential forms and the q-Tate twist of the constant sheaf, respectively.

In [Kat89, p. 1.4] Kato builds lifting maps from the cohomology groups $\mathrm{H}^q(F, \mathbb{Z}/p(q-1))$ to their characteristic zero counterparts, $\mathrm{H}^q(M^h, \mathbb{Z}/p(q-1))$, denoted as

$$\iota^q : \mathrm{H}^q(F, \mathbb{Z}/p(q-1)) \to \mathrm{H}^q(M^h, \mathbb{Z}/p(q-1)).$$

Using the Kummer map $(M^h)^\times \to \mathrm{H}^1(M^h, \mathbb{Z}/p(1))$ and the cup product, we define a product

$$\mathrm{H}^q(M^h, \mathbb{Z}/p(q-1)) \times (M^h)^\times \to \mathrm{H}^{q+1}(M^h, \mathbb{Z}/p(q))$$
$$(\chi, a) \mapsto \{\chi, a\}.$$

By [Kat89, p. 1.4], the map

$$\lambda_\pi\colon \mathrm{H}^q(F,\mathbb{Z}/p(q-1))\oplus \mathrm{H}^{q-1}(F,\mathbb{Z}/p(q-2))\to \mathrm{H}^q(M^h,\mathbb{Z}/p(q-1))$$
$$(\chi,\psi)\mapsto \iota^q(\chi)+\{\iota^{q-1}(\psi),\pi\}$$

is injective. Moreover, by [Kat89, p. 1.3], there is a surjective map

$$\delta\colon \Omega_F^{q-1}\to \mathrm{H}^q(F,\mathbb{Z}/p(q-1)),$$

which allows one to represent elements in $\mathrm{H}^q(F,\mathbb{Z}/p(q-1))$ using differential forms.

By [BN23, Lemma 2.13], if $\phi : M^h \to \tilde{M}^h$ is a finite homomorphism of henselian fields over $\mathbb{Q}_p$ with ramification index e and $\tilde{M}^h$ has a uniformiser $\tilde{\pi}$ which is algebraic over $\mathbb{Q}_p$, one has

$$\phi_*\lambda_\pi(\delta(\alpha),\delta(\beta)) = \lambda_{\tilde{\pi}}(\delta(\phi_*\alpha), e\delta(\phi_*\beta)),\quad \text{in } \mathrm{H}^q(\tilde{M}^h,\mathbb{Z}/p(q-1)). \tag{2.1.3}$$

for every $\alpha,\beta\in \Omega_F^{q-1}\oplus\Omega_F^{q-2}$.

2.1.2. *Definition of* fil_0 *and* ∂. We now go back to our setting, in which M^h is $K(X)^h$ and the residue field F is $k(Y)$, the function field of the special fibre Y. The subgroup $\mathrm{fil}_0(K(X)^h)\subseteq \mathrm{Br}(K(X)^h)[p]\simeq \mathrm{H}^2(K(X)^h,\mathbb{Z}/p(1))$ is then defined[1] as

$$\mathrm{fil}_0(K(X)^h,\mathbb{Z}/p(1)) := \mathrm{Im}(\lambda_\pi\colon \mathrm{H}^2(k(Y),\mathbb{Z}/p(1))\oplus \mathrm{H}^1(k(Y),\mathbb{Z}/p)\to \mathrm{H}^2(K(X)^h,\mathbb{Z}/p(1))).$$

Since λ_π is injective, we can then define the residue map

$$\partial := p_2\circ\lambda_\pi^{-1}\colon\ \mathrm{fil}_0\,\mathrm{H}^2(K(X)^h)\to \mathrm{H}^1(k(Y),\mathbb{Z}/p)$$

which is the projection onto $\mathrm{H}^1(k(Y),\mathbb{Z}/p)$ of the inverse of λ_π. Its restriction to $\mathrm{fil}_0(X)\subseteq \mathrm{Br}(X)[p]$ induces (see [BN23, Proposition 3.1]) the required map

$$\partial\colon\ \mathrm{fil}_0(X)\to \mathrm{H}^1(Y,\mathbb{Z}/p).$$

2.1.3. *Definition of* fil_n *and* $\mathrm{rsw}_{n,\pi}$. Let L be the field of fraction of $\mathcal{O}^h_{\mathcal{X},Y}[T]_{(\pi)}$. Again, using the Kummer map $L^\times\to \mathrm{H}^1(L,\mathbb{Z}/p(1))$, the cup product and the natural map $\mathrm{H}^2(K(X)^h,\mathbb{Z}/p(1))\to \mathrm{H}^2(L,\mathbb{Z}/p(1))$, we define a product

$$\mathrm{H}^2(K(X)^h,\mathbb{Z}/p(1))\times L^\times \to \mathrm{H}^3(L,\mathbb{Z}/p(2))$$
$$(\chi,a)\mapsto\{\chi,a\}.$$

The subgroup $\mathrm{fil}_n(K(X)^h)\subseteq \mathrm{Br}(K(X)^h)[p]\simeq \mathrm{H}^2(K(X)^h,\mathbb{Z}/p(1))$ is then defined as

$$\mathrm{fil}_n(K(X)^h) := \{\chi\in \mathrm{H}^2(K(X)^h,\mathbb{Z}/p(1)) \text{ such that } \{\chi,1+\pi^{n+1}T\}=0 \text{ in } \mathrm{H}^3(L,\mathbb{Z}/p(2))\}.$$

By [Kat89, Section 5] together with [Kat89, Proposition 6.3], for every $n\geq 1$ and $\chi\in \mathrm{fil}_n(K(X)^h)$, there exists a unique pair (α,β) in $\Omega^2_{k(Y)}\oplus\Omega^1_{k(Y)}$ such that

$$\{\chi,1+\pi^nT\} = \lambda_\pi(\delta(T\alpha),\delta(T\beta)),\quad \text{in } \mathrm{H}^3(L,\mathbb{Z}/p(2)). \tag{2.1.4}$$

The pair (α,β) is called the refined Swan conductor of $\chi\in \mathrm{fil}_n(K(X)^h)\subseteq \mathrm{H}^2(K(X)^h,\mathbb{Z}/p(1))$. Hence, for every $n\geq 1$, we get an homomorphism

$$\mathrm{rsw}_{n,\pi}\colon\ \mathrm{fil}_n(K(X)^h)\to \Omega^2_{k(Y)}\oplus\Omega^1_{k(Y)},$$

[1] Kato's original definition of fil_0 is different; however, he proves in [Kat89, Proposition 6.1] that the definition we gave here is equivalent to his. Finally, he also proves that fil_0 can be realised as the kernel of the natural map $\mathrm{H}^2(K(X)^h,\mathbb{Z}/p(1))\to \mathrm{H}^2(K(X)^{sh},\mathbb{Z}/p(1))$, which gives back the description of fil_0 provided in equation (2.1.1).

whose restriction to $\mathrm{fil}_n(X) \subseteq \mathrm{Br}(X)[p]$, induces (see [BN23, Theorem 8.1.(1)]) the required map

$$\mathrm{rsw}_{n,\pi}\colon\ \mathrm{fil}_n(X) \to \mathrm{H}^0(Y, \Omega^2) \oplus \mathrm{H}^0(Y, \Omega^1).$$

2.2. **Proof of Theorem 1.3.1.** Choosing an isomorphism between $\mathbb{Z}/p$ and $\mathbb{Z}/p(1)$ over $\overline{K}$, we may and do replace $\mathbb{Z}/p$ with $\mathbb{Z}/p(1)$ in the statement of Theorem 1.3.1. Since elements of $\mathrm{NS}(X_{\overline{K}})$ are already zero when restricted to $\overline{K}(X)$, the restriction map

$$\mathrm{H}^2(X_{\overline{K}}, \mathbb{Z}/p(1)) \to \mathrm{H}^2(\overline{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p(1))$$

factors through a map

$$g : \mathrm{Br}(X_{\overline{K}})[p] \to \mathrm{H}^2(\overline{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p(1)).$$

We are going to prove that $\mathrm{Ker}(g) = \mathrm{Br}(X_{\overline{K}})^{gb}[p]$ by double inclusion.

2.2.1. $\subseteq$. First we prove the inclusion $\mathrm{Ker}(g) \subseteq \mathrm{Br}(X_{\overline{K}})^{gb}[p]$. Let $\widetilde{\mathcal{A}} \in \mathrm{Ker}(g)$ and let L be a finite extension of K such that there exists $\mathcal{A} \in \mathrm{Br}(X_L)[p]$ with $\mathcal{A}_{\overline{K}} = \widetilde{\mathcal{A}}$. Since $\widetilde{\mathcal{A}} \in \mathrm{Ker}(g)$, upon replacing L with a finite extension, we can assume that $\mathcal{A}$ is in the kernel of $g_L\colon\ \mathrm{Br}(X_L)[p] \to \mathrm{Br}(L(X)^{\mathrm{sh}})[p]$, i.e. in $\mathrm{fil}_0(X_L)$, cf. (2.1.1). By the following Proposition 2.2.1, upon replacing L with a finite field extension, we can assume that $\partial(\mathcal{A}) = 0$. By 2.1.2(1), this implies that $\mathcal{A} \in \mathrm{Br}(\mathcal{X}_L)[p] \subseteq \mathrm{Br}(X_L)[p]$. But then, since $\mathcal{X}(\mathcal{O}_L) = X(L)$, the evaluation map factors through $\mathrm{Br}(\mathcal{O}_L) = 0$ and hence it is constantly zero.

Proposition 2.2.1. *Let $\mathcal{A} \in \mathrm{Br}(X)$. There exists a finite extension K'/K such that $\mathcal{A}_{K'} \in \mathrm{fil}_0\,\mathrm{Br}(X_{K'})$ if and only if the image of $\mathcal{A}$ in $\mathrm{Br}(K(X)^{\mathrm{h}})$ can be written as a sum $\mathcal{B} + \mathcal{C}$ where $\mathcal{B} \in \mathrm{fil}_0\,\mathrm{Br}(K(X)^{\mathrm{h}})$ has residue zero and $\mathcal{C} \in \mathrm{Br}(K(X)^{\mathrm{h}})$ is such that $\mathcal{C}_{K(X)^{\mathrm{h}}L} = 0$ for some finite extension L/K.*

Proof. The if implication follows from the fact that $\mathrm{fil}^0(K(X)^{\mathrm{h}})$ maps to $\mathrm{fil}^0(L)$ for every finite extension $K(X)^{\mathrm{h}} \subseteq L$, thanks to [BN23, Lemma 2.16].

For the forwards implication, suppose K'/K is a finite extension with ramification index ε and uniformiser π' such that $\mathcal{A}_{K'} \in \mathrm{fil}_0(X_{K'})$. Let $k(Y)'$ be the residue field of $K(X)^{\mathrm{h}}K'$. By definition of fil_0 (see Section 2.1.2), there exist $(\chi, \psi) \in \mathrm{H}^2(k(Y)', \mathbb{Z}/p(1)) \oplus \mathrm{H}^1(k(Y)', \mathbb{Z}/p)$ such that

$$\mathcal{A}_{K(X)^{\mathrm{h}}K'} = \lambda_{\pi'}(\chi, \psi).$$

Let $\sigma \in \Gamma_k$ be such that its image $\bar{\sigma} \in \mathrm{Gal}(k(Y)'/k(Y))$ generates $\mathrm{Gal}(k(Y)'/k(Y))$. Since $\mathcal{A}^{\sigma} = \mathcal{A}$ and $\lambda_{\pi'}$ is injective, we have $\chi^{\sigma} = \chi \in \mathrm{Br}(k(Y)')$ (and $\psi^{\sigma} = \psi$). Thus by the following Lemma 2.2.2, there exists a $\chi_0 \in \mathrm{Br}(k(Y))$ such that $\mathrm{res}_{k(Y)'/k(Y)}\chi_0 = \chi$. Let $\mathcal{B} = \lambda_{\pi}(\chi_0, 0) \in \mathrm{fil}_0(K(X)^{\mathrm{h}})$. Note that $\mathcal{B}$ has residue zero by definition (see Section 2.1.2), hence $\mathcal{B}$ has zero residue on every finite extension. it remains to prove that $\mathcal{C} := \mathcal{A}_{K(X)^{\mathrm{h}}} - \mathcal{B}$ vanishes after a finite extension. Applying (2.1.3), with $\phi : K(X)^h \subseteq K(X)^h K'$ being the natural inclusion, shows that $\mathcal{B}_{K(X)^{\mathrm{h}}K'} = \lambda_{\pi'}(\chi, 0)$ so that

$$\mathcal{C}_{K(X)^{\mathrm{h}}K'} = \lambda_{\pi'}(\chi, \psi) - \lambda_{\pi'}(\chi, 0) = \lambda_{\pi'}(0, \psi) = \{\iota^1(\psi), \pi'\}$$

which is split by adjoining a pth root of π'. □

Lemma 2.2.2. *Let L/F be a Galois extension of fields such that $G := \mathrm{Gal}(L/F)$ is cyclic. Then the natural map $\mathrm{Br}(F) \to (\mathrm{Br}(L))^G$ is surjective.*

Proof. By looking at the Hochschild–Serre spectral sequence

$$E_2^{a,b} := \mathrm{H}^a(G, \mathrm{H}^b(L, \mathbb{G}_m)) \implies \mathrm{H}^{a+b}(F, \mathbb{G}_m),$$

it is enough to show that $E_2^{0,2} = E_\infty^{0,2}$. For this, we prove that $E_2^{2,1} = E_2^{3,0} = 0$. Since $\mathrm{Pic}(L) = 0$, one has $E_2^{2,1} = \mathrm{H}^2(G, \mathrm{Pic}(L)) = 0$. On the other hand, since G is cyclic and using Hilbert's Theorem 90, one has $E_2^{3,0} = \mathrm{H}^3(G, L^*) = \mathrm{H}^1(G, L^\times) = 0$. □

2.2.2. $\supseteq$. Now we prove the inclusion $\mathrm{Br}(X_{\overline{K}})^{gb}[p] \subseteq \mathrm{Ker}(g)$. Let $\widetilde{\mathcal{A}} \in \mathrm{Br}(X_{\overline{K}})[p] \backslash \mathrm{Ker}(g)$ and let L be a finite extension of K such that there exists $\mathcal{A} \in \mathrm{Br}(X_L)[p]$ with $\mathcal{A}_{\overline{K}} = \widetilde{\mathcal{A}}$. Since $\widetilde{\mathcal{A}} \notin \mathrm{Ker}(g)$, upon replacing L with a finite extension, we can assume that for every extension $L \subseteq F$ one has $g_F(\mathcal{A}) \neq 0$ where $g_F : \mathrm{Br}(X_F)[p] \to \mathrm{Br}(F(X)^{\mathrm{sh}})[p]$ is the natural map, i.e. $\mathcal{A} \notin \mathrm{fil}_0(X_F)$. By the following Proposition 2.2.3, upon replacing again L with a finite field extension, we get that $\widetilde{\mathcal{A}} \notin \mathrm{Br}(X_{\overline{K}})^{gb}[p]$.

Proposition 2.2.3. *Let $\mathcal{A} \in \mathrm{Br}(X)[p]$ and assume that $\mathcal{A} \notin \mathrm{fil}_0(X_F)$ for all finite extensions F of K. Let k'/k be a finite extension such that for every non-zero $(\alpha, \beta) \in \mathrm{H}^0(Y, \Omega^2) \oplus \mathrm{H}^0(Y, \Omega^1)$ there exists $P \in Y(k')$ such that $(\alpha_P, \beta_P) \neq 0$ and K'/K the corresponding unramified extension. Then, for every extension L/K' the evaluation map $\mathrm{ev}_{\mathcal{A}}^L : X(L) \to \mathrm{Br}(L)[p]$ is surjective.*

Proof. By Theorem 2.1.2(2) it is enough to show that for every L/K' there exists $n \geq 1$ such that $\mathcal{B} \in \mathrm{fil}_n(X_L)$ with $\mathrm{rsw}_{n,\pi_L}(\mathcal{B}) = (\alpha, \beta) \in \mathrm{H}^0(Y_{k_L}, \Omega^2) \oplus \mathrm{H}^0(Y_{k_L}, \Omega^1)$ and such that there exists $P \in Y_{k_L}(k_L)$ with $(\alpha_P, \beta_P) \neq (0,0)$. By assumption, we know that $\mathcal{A}_L \notin \mathrm{fil}_0(X_L)$, hence there exists $n \geq 1$ such that $\mathcal{A}_L \in \mathrm{fil}_n(X_L) \backslash \mathrm{fil}_{n-1}(X_L)$ so that $\mathrm{rsw}_{n,\pi_L}(\mathcal{A}_L) \neq (0,0)$. By Lemma 2.2.4 below, we have

$$\mathrm{rsw}_{n,\pi_L}(\mathcal{A}_L) = \bar{c}^n(\alpha_0, \beta_0)$$

with $\bar{c}$ a unit in the residue field k_L of $\mathcal{O}_L$ and $(\alpha_0, \beta_0) \in \mathrm{H}^0(Y, \Omega^2) \oplus \mathrm{H}^0(Y, \Omega^1)$. By assumption, there exists $P \in Y(k') \subseteq Y(k_L)$ such that $(\bar{c}^n \alpha_{0,P}, \bar{c}^n \beta_{0,P}) \neq (0,0)$ and hence $\mathrm{ev}_{\mathcal{A}}^L$ is non-constant on L-points. This concludes the proof. □

Lemma 2.2.4. *Let L/K be a finite field extension with ring of integers $\mathcal{O}_L$ and residue field k_L. Fix a uniformiser π_L of $\mathcal{O}_L$.*

(1) Let $\mathcal{A}_L \in \mathrm{fil}_n(X_L)$ for $n \geq 1$. Write $\mathrm{rsw}_{n,\pi_L}(\mathcal{A}_L) = (\alpha, \beta)$. Then for $\sigma \in \mathrm{Gal}(L/K)$,

$$\mathrm{rsw}_{n,\pi_L}(\sigma_*(\mathcal{A}_L)) = \bar{a}^{-n} \cdot (\sigma_*(\alpha), \sigma_*(\beta))$$

with $\bar{a}$ the image in k_L of $a := \sigma(\pi_L)/\pi_L$.

(2) Let $\mathcal{A} \in \mathrm{Br}(X)[p]$ be such that $\mathcal{A}_L \in \mathrm{fil}_n(X_L)$ for $n \geq 1$. Then

$$\mathrm{rsw}_{n,\pi_L}(\mathcal{A}_L) = \bar{c}^n(\alpha_0, \beta_0)$$

with $\bar{c} \in k_L^\times$ and $(\alpha_0, \beta_0) \in \mathrm{H}^0(Y, \Omega^2) \oplus \mathrm{H}^0(Y, \Omega^1)$.

Proof.

(1) Let R_L denote the henselisation of the local ring $\mathcal{O}_{\mathcal{X}_{\mathcal{O}_L}, Y_{k_L}}$. An element $\sigma \in \mathrm{Gal}(L/K)$ induces a ring morphism $\sigma \colon R_L[T] \to R_L[T]$. By definition of rsw_{n,π_L}, it is enough to show that

$$\lambda_{\pi_L}(\delta(\bar{a}^{-n} T \sigma_*(\alpha)), \delta(\bar{a}^{-n} T \sigma_*(\beta))) = \{\sigma_* \mathcal{A}_L, 1 + \pi_L^n T\}.$$

Let $\tilde{\phi}\colon (R_L[T])^h_{(\pi_L)} \to (R_L[T])^h_{(\pi_L)}$ be the map sending T to $a^{-n}T$ and write $\phi := \tilde{\phi}\circ\sigma$, so that

$$(\bar{a}^{-n}T\sigma_*(\alpha), \bar{a}^{-n}T\sigma_*(\beta)) = (\phi_*(\alpha T), \phi_*(\beta T)).$$

Applying (2.1.3) with $\phi\colon (R_L[T])^h_{(\pi_L)} \to (R_L[T])^h_{(\pi_L)}$,

$$\begin{aligned}\lambda_{\pi_L}(\delta(\bar{a}^{-n}T\sigma_*(\alpha)), \delta(\bar{a}^{-n}T\sigma_*(\beta))) &= \phi_*(\lambda_{\pi_L}(\delta(\alpha T), \delta(\beta T)))\\ &= \phi_*(\{\mathcal{A}_L, 1+\pi_L^n T\}),\end{aligned}$$

where the first equality comes from the fact that the ramification index of ϕ is 1, while the second is the definition of refined Swan conductor. The proof is concluded by observing that

$$\phi_*(\{\mathcal{A}_L, 1+\pi_L^n T\}) = \{\sigma_*(\mathcal{A}_L), 1+\sigma(\pi_L)^n a^{-n}T\} = \{\sigma_*(\mathcal{A}_L), 1+\pi_L^n T\}.$$

(2) Let $(\alpha,\beta) := \mathrm{rsw}_{n,\pi_L}(\mathcal{A}_L)$ and let e be the ramification index of L/K. Write $c := \left(\frac{\pi^e}{\pi_L}\right)^n$ and define

$$\alpha_0 := c\alpha \quad \text{and} \quad \beta_0 := c\beta,$$

so that it is enough to show that α_0 and β_0 are Galois invariant. The element $\mathcal{A}_L$ is Galois invariant by construction, hence for any $\sigma \in \mathrm{Gal}(L/K)$

$$(\alpha,\beta) = \mathrm{rsw}_{n,\pi_L}(\mathcal{A}_L) = \mathrm{rsw}_{n,\pi_L}(\sigma_*(\mathcal{A}_L)) = \bar{a}^{-n}(\sigma_*(\alpha), \sigma_*(\beta)),$$

where the last equality follows from part (1). Hence, $\bar{a}^{-n}\sigma_*(\alpha) = \alpha$ so that,

$$\sigma_*(\alpha_0) = \left(\frac{\sigma(\pi)^e}{\sigma(\pi_L)}\right)^n \sigma_*(\alpha) = \frac{((\pi)^e)^n}{\sigma(\pi_L)^n}\frac{\sigma(\pi_L)^n}{\pi_L^n}\alpha = \left(\frac{\pi^e}{\pi_L}\right)^n \alpha = \alpha_0,$$

whereby α_0 is Galois invariant. The same argument applies to β_0 and concludes the proof. □

2.3. ***p*-adic vanishing cycles.** To be able to reinterpret Theorem 1.3.1 in terms of prismatic cohomology, we need to reinterpret Theorem 1.3.1 in terms of the p-adic vanishing cycles spectral sequence, following [BN23, Proof of Theorem C]. Let $j : X \to \mathcal{X}$ be the natural open immersion. Recall that there is a spectral sequence

$$E_2^{a,b} := \mathrm{H}^a(\mathcal{X}_{\mathcal{O}_{\overline{K}}}, R^b j_*\mathbb{Z}/p) \Rightarrow \mathrm{H}^{a+b}(X_{\overline{K}}, \mathbb{Z}/p).$$

We consider the the edge map

$$\mathrm{H}^n(X_{\overline{K}}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\overline{K}}}, R^n j_*\mathbb{Z}/p)$$

and its variant over $\mathcal{O}_{\mathbb{C}_p}$

$$\mathrm{H}^n(X_{\mathbb{C}_p}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, R^n j_*\mathbb{Z}/p).$$

Corollary 2.3.1. *One has an equality*

$$\mathrm{Ker}(\mathrm{H}^n(X_{\overline{K}}, \mathbb{Z}/p) \to \mathrm{H}^n(\overline{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p)) = \mathrm{Ker}(\mathrm{H}^n(X_{\mathbb{C}_p}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, R^n j_*\mathbb{Z}/p)),$$

and an isomorphism

$$\mathrm{Br}(X_{\overline{K}})[p]/\,\mathrm{Br}(X_{\overline{K}})^{\mathrm{g}b}[p] = \mathrm{Im}(\mathrm{H}^2(X_{\mathbb{C}_p}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, R^2 j_*\mathbb{Z}/p)).$$

Proof. The second part follows from the first part and Theorem 1.3.1. For the first part, write $\overline{K}(X)^{\mathrm{h}}$ for the henselisation of $\overline{K}(X)$, so that $\overline{K}(X)^{\mathrm{sh}}$ is the maximal unramified extension of $\overline{K}(X)^{\mathrm{h}}$. Let $\Gamma := \mathrm{Gal}(\overline{K}(X)^{\mathrm{sh}}/\overline{K}(X)^{\mathrm{h}})$. Consider the Hochschild–Serre spectral sequence for $\overline{K}(X)^{\mathrm{h}} \subseteq \overline{K}(X)^{\mathrm{sh}}$,

$$E_2^{a,b} := \mathrm{H}^a(\Gamma, \mathrm{H}^b(\overline{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p)) \Rightarrow \mathrm{H}^{a+b}(\overline{K}(X)^{\mathrm{h}}, \mathbb{Z}/p).$$

The edge maps

$$\mathrm{H}^n(X_{\overline{K}}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\overline{K}}}, R^n j_*\mathbb{Z}/p) \quad \text{and} \quad \mathrm{H}^n(\overline{K}(X)^{\mathrm{h}}, \mathbb{Z}/p) \to \mathrm{H}^0(\Gamma, \mathrm{H}^n(\overline{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p)))$$

fit into a commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^n(X_{\overline{K}}, \mathbb{Z}/p) & \longrightarrow & \mathrm{H}^n(\overline{K}(X)^{\mathrm{h}}, \mathbb{Z}/p) & & \\
\downarrow & & \downarrow & & \\
\mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\overline{K}}}, R^n j_*\mathbb{Z}/p) & \xrightarrow{g} & \mathrm{H}^0(\Gamma, \mathrm{H}^n(\overline{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p))) & \hookrightarrow & \mathrm{H}^n(\overline{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p)),
\end{array}$$

in which the horizontal arrows in the square are the natural restriction maps and bottom right map is the natural inclusion. By [BN23, Lemma 3.4], the map g is injective, hence

$$\mathrm{Ker}(\mathrm{H}^n(X_{\overline{K}}, \mathbb{Z}/p) \to \mathrm{H}^n(\overline{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p)) = \mathrm{Ker}(\mathrm{H}^n(X_{\overline{K}}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\overline{K}}}, R^n j_*\mathbb{Z}/p)).$$

Since étale cohomology is invariant under algebraically closed field extensions, one has

$$\mathrm{Ker}(\mathrm{H}^n(X_{\overline{K}}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\overline{K}}}, R^n j_*\mathbb{Z}/p)) \simeq \mathrm{Ker}(\mathrm{H}^n(X_{\mathbb{C}_p}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, R^n j_*\mathbb{Z}/p)),$$

and this concludes the proof. □

3. Prismatic cohomology and p-adic vanishing cycles

In this section we prove Theorem 1.3.2. We start with Section 3.1, in which we recall results and definitions on $\mathcal{O}_{\mathbb{C}_p^\flat}$, A_{inf} and prismatic cohomology from [BS22]. With these in our hands, in Section 3.2 we perform some computations in the mod-p version of prismatic cohomology, that we will use in the rest of the paper. Finally, in Section 3.3 we prove Theorem 1.3.2.

In this section k denotes the residue field of $\mathcal{O}_{\mathbb{C}_p}$. Let $\mathcal{X}$ be a smooth proper scheme over $\mathcal{O}_{\mathbb{C}_p}$ and $\widehat{\mathcal{X}}$ its formal p-adic completion. We write Y for $\mathcal{X}_k$ and X for $\mathcal{X}_{\mathbb{C}_p}$. Let

$$h^{a,b} := \dim_{\mathbb{C}_p}(\mathrm{H}^b(X, \Omega^a_{X/\mathbb{C}_p})) \quad \text{and} \quad h^n := \dim_{\mathbb{C}_p}(\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C}_p)) = \dim_{\mathbb{Q}_p}(\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_p)).$$

3.1. **Recollection on prismatic cohomology.**

3.1.1. *Tilt.* A reference for what follows is [Bha17, Sections 2 and 3]. Let $\mathcal{O}_{\mathbb{C}_p^\flat}$ be the tilt of $\mathcal{O}_{\mathbb{C}_p}$. Recall that

$$\mathcal{O}_{\mathbb{C}_p^\flat} \simeq \varprojlim_{\varphi_{\mathcal{O}_{\mathbb{C}_p}/p}} \mathcal{O}_{\mathbb{C}_p}/p \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p},$$

where the first isomorphism is as rings and the second as multiplicative monoids, and the first limit is done along the power of the absolute Frobenius $\varphi_{\mathcal{O}_{\mathbb{C}_p}/p} : \mathcal{O}_{\mathbb{C}_p}/p \to \mathcal{O}_{\mathbb{C}_p}/p$ and the second along the p^{th}-power map $(-)^p : \mathcal{O}_{\mathbb{C}_p} \to \mathcal{O}_{\mathbb{C}_p}$.

The choice of a compatible system $\{\zeta_{p^n}\}$ of primitive p-power roots of unity in $\mathcal{O}_{\mathbb{C}_p}$ gives, via the isomorphism $\mathcal{O}_{\mathbb{C}_p^\flat} \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$, an element

$$\epsilon := (1, \zeta_p, \ldots, \zeta_{p^n}, \ldots) \in \mathcal{O}_{\mathbb{C}_p^\flat}.$$

The p-adic valuation of $\mathcal{O}_{\mathbb{C}_p}$ induces, precomposing with the projection on the first component, a rank 1 valuation v on $\mathcal{O}_{\mathbb{C}_p^\flat}$, such that the element

$$d := \sum_{i=0}^{p-1} \epsilon^{i/p}, \tag{3.1.1}$$

has $v(d) = 1$. Then v makes $\mathcal{O}_{\mathbb{C}_p^\flat}$ a complete valuation ring, whose maximal ideal $\mathfrak{m}^\flat$ is generated by d^α for $\alpha \in \mathbb{Q}_{>0}$.

3.1.2. A_{inf}. A reference for what follows is [BMS18, Section 3, Example 3.16]. Set $A_{\mathrm{inf}} := W(\mathcal{O}_{\mathbb{C}_p^\flat})$ and write $\varphi : A_{\mathrm{inf}} \to A_{\mathrm{inf}}$ for the Frobenius automorphism. If $x \in \mathcal{O}_{\mathbb{C}_p^\flat}$, we denote by $[x]$ its Teichmuller lift. Recall that there is a natural map $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$, whose kernel is principal generated by

$$\xi := \sum_{i=0}^{p-1} [\epsilon^{i/p}]$$

whose reduction modulo p is d.

Since $\mathrm{Ker}(\theta) = (\xi)$ is principal, for every $n \geq 1$, sending 1 to ξ^n and d^n induces natural isomorphisms

$$A_{\mathrm{inf}} \to A_{\mathrm{inf}} \otimes (\xi^n) \quad \text{and} \quad \mathcal{O}_{\mathbb{C}_p^\flat} \to \mathcal{O}_{\mathbb{C}_p^\flat} \otimes (d^n)$$

respectively. If M is an A_{inf}-module (resp. $\mathcal{O}_{\mathbb{C}_p^\flat}$-module), we will identify $M \otimes (\xi^n)$ (resp. $M \otimes (d^n)$) with M along this isomorphism, and if $f : N \to M \otimes (\xi^n)$ (resp. $f : N \to M \otimes (d^n)$) is a morphisms of A_{inf}-module (resp. $\mathcal{O}_{\mathbb{C}_p^\flat}$-module), we will write $f_n : N \to M$ for the induced morphism.

We will work mainly with the following commutative diagram

$$\begin{array}{ccc}
 & & \mathcal{O}_{\mathbb{C}_p} \\
 & \overset{\widetilde{\theta}}{\nearrow} & \uparrow{\scriptstyle\theta} \\
A_{\mathrm{inf}} & \xrightarrow{\varphi} & A_{\mathrm{inf}} \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\
W(k) & \xrightarrow{\varphi} & W(k),
\end{array}$$

where β is induced by the projection $\mathcal{O}_{\mathbb{C}_p^\flat} \to k$ via the Witt vectors functoriality and $\widetilde{\theta} := \theta \circ \varphi^{-1}$, and its modulo p version

$$\begin{array}{ccc}
 & & \mathcal{O}_{\mathbb{C}_p}/p \simeq \mathcal{O}_{\mathbb{C}_p^\flat}/d \\
 & \overset{\widetilde{\theta}}{\nearrow} & \uparrow{\scriptstyle\theta} \\
\mathcal{O}_{\mathbb{C}_p^\flat} & \xrightarrow{\varphi} & \mathcal{O}_{\mathbb{C}_p^\flat} \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\
k & \xrightarrow{\varphi} & k.
\end{array}$$

3.1.3. *Some sheaves on the prismatic sites.* A reference for what follows is [BS22, Section 4]. Consider the prism $\Delta := (A_{\mathrm{inf}}, (\xi))$. Let $\mathrm{Sh}_{\Delta}(\mathcal{X})$ be the category of sheaves in the prismatic site of $\widehat{\mathcal{X}}$ (with the flat topology, see [BS22, Section 4.1]). There are two interesting sheaves of A_{inf}-modules:

(1) $\mathcal{O}_{\mathcal{X}/\Delta}$, which sends (B, J) to B
(2) $\overline{\mathcal{O}}_{\mathcal{X}/\Delta}$, which sends (B, J) to B/J.

Let $\mathrm{Sh}_{\text{ét}}(\widehat{\mathcal{X}})$ be the category of sheaves in the étale site of $\widehat{\mathcal{X}}$. There is a natural functor (see [BS22, Construction 4.4])

$$\nu_* : \mathrm{Sh}_{\text{ét}}(\widehat{\mathcal{X}}) \to \mathrm{Sh}_{\Delta}(\mathcal{X}).$$

Let

$$\Delta := R\nu_*\mathcal{O}_{\mathcal{X}/\Delta} \quad \text{and} \quad \overline{\Delta} := R\nu_*\overline{\mathcal{O}}_{\mathcal{X}/\Delta}.$$

These are complexes of sheaves of A_{inf} and $\mathcal{O}_{\mathbb{C}_p}$-modules, respectively, such that

$$\Delta \otimes^L \mathcal{O}_{\mathbb{C}_p} \simeq \overline{\Delta}$$

and Δ is endowed with a natural $\varphi_{A_{\mathrm{inf}}}$-linear Frobenius $\varphi : \Delta \to \Delta$.

Set

$$\Delta^{(1)} := \varphi^*_{A_{\mathrm{inf}}}\Delta,$$

so that the $\varphi_{A_{\mathrm{inf}}}$-linear Frobenius $\varphi : \Delta \to \Delta$, induces an A_{inf}-linear map $\phi : \Delta^{(1)} \to \Delta$ and $\varphi_{A_{\mathrm{inf}}}$-linear map $\varphi^{(1)} : \Delta^{(1)} \to \Delta^{(1)}$, fitting into a commutative diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\varphi} & \Delta \\ {\scriptstyle s}\downarrow & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle s} \\ \Delta^{(1)} & \xrightarrow[\varphi^{(1)}]{} & \Delta^{(1)} \end{array}$$

where s is the natural inclusion.

By [BS22, Theorems 15.2 and 15.3] (see also [TZ23, Construction 5.26] for a global statement in a more general situation) there exists a decreasing filtration, called the Nygaard filtration,

$$\iota : N^{\geq i} \to \Delta^{(1)}$$

and a A_{inf}-linear map $\phi_i : N^{\geq i} \to \Delta$, called the i^{th}-divided Frobenius, making the following diagram commutative

$$N^{\geq i} \xrightarrow{\phi_i} \Delta \xrightarrow{\xi^i} \Delta, \qquad \xi^i \circ \phi_i = \phi \circ \iota.$$

We write $\varphi_i^{(1)} : N^{\geq i} \to \Delta^{(1)}$ for the $\varphi_{A_{\mathrm{inf}}}$-linear compositum $s \circ \phi_i$.

3.1.4. *Comparison isomorphisms.* The following summarises the main results we need from [BS22].

Theorem 3.1.2. *There are exact triangles in $D^b(\widehat{\mathcal{X}}, \mathcal{O}_{\mathbb{C}_p^\flat})$*

$$\tau_{\leq i-1}(\overline{\Delta}/p) \to \tau_{\leq i}(\overline{\Delta}/p) \to \Omega^i_{\mathcal{X}/p/(\mathcal{O}_{\mathbb{C}_p}/p)} \tag{3.1.3}$$

$$N^{\geq i+1}/p \to N^{\geq i}/p \xrightarrow{\phi_i} \tau_{\leq i}(\overline{\Delta}/p) \tag{3.1.4}$$

and an isomorphism

$$\varphi^*\overline{\Delta}/p \simeq \Omega^\bullet_{\mathcal{X}/p/\mathcal{O}_{\mathbb{C}_p}/p} \tag{3.1.5}$$

Proof. Since $\mathcal{X}$ is smooth, $\Omega^j_{\mathcal{X}/\mathcal{O}_{\mathbb{C}_p}}$ is locally free. Hence (3.1.3) follows applying $-\otimes^L \mathbb{Z}/p$ to the Hodge-Tate comparison from [BS22, Theorem 4.11] thanks to the subsequent Lemma 3.1.6(1). Similarly, (3.1.4) follows from [BS22, Theorem 15.3] (see also [TZ23, Corollary 5.27] for a global statement) after applying $-\otimes^L \mathbb{Z}/p$ thanks to the subsequent Lemma 3.1.6(1). Finally, (3.1.5) follows from the de Rham comparison [BS22, Theorem 6.4] after applying $-\otimes^L \mathbb{Z}/p$ □

Lemma 3.1.6. *Let R be a ring $\mathcal{F}^\bullet \in D^b(\widehat{\mathcal{X}}, R)$. Assume that $\mathcal{H}^n(\mathcal{F}^\bullet)$ is finite locally free over R for every n. Then:*

(1) For every ring morphism $R \to S$, there is a natural isomorphism $(\tau_{\le i}\mathcal{F}^\bullet)\otimes^L S \simeq \tau_{\le i}(\mathcal{F}^\bullet \otimes^L S)$.

(2) Assume that R is coherent complete local ring with residue field k and that $\mathrm{H}^i(\mathcal{H}^n(\mathcal{F}^\bullet))$ and $\mathrm{H}^i(\mathcal{F}^\bullet)$ are finite free for every i and n. If the conjugate spectral sequence

$${}_kE_2^{a,b} := \mathrm{H}^a(\mathcal{H}^b(\mathcal{F}^\bullet \otimes^L k)) \Rightarrow \mathrm{H}^{a+b}(\mathcal{F}^\bullet \otimes^L k)$$

for $\mathcal{F}^\bullet \otimes^L k$ degenerates at the second page, then the conjugate spectral sequence

$$E_2^{a,b} := \mathrm{H}^a(\mathcal{H}^b(\mathcal{F}^\bullet)) \Rightarrow \mathrm{H}^{a+b}(\mathcal{F}^\bullet)$$

for $\mathcal{F}^\bullet$ degenerates at the second page.

Proof.

(1) Assume that $\mathcal{F}^\bullet$ is concentrated in degree $\le r$. We prove the statement by decreasing induction on i, the case $i = r+1$ being clear since both sides identify with $\mathcal{F}^\bullet \otimes^L S$. Assume now $i < r+1$ and consider the exact triangle

$$\tau_{\le i}F^\bullet \to \tau_{\le i+1}F^\bullet \to \mathcal{H}^i(F^\bullet),$$

giving an exact triangle

$$(\tau_{\le i}F^\bullet)\otimes^L S \to (\tau_{\le i+1}F^\bullet)\otimes^L S \to (\mathcal{H}^i(F^\bullet))\otimes^L S.$$

By induction $(\tau_{\le i+1}F^\bullet)\otimes^L S \simeq \tau_{\le i+1}(F^\bullet \otimes^L S)$. Since $(\tau_{\le i}F^\bullet)\otimes^L S$ is concentrated in degree $\le i$, we get a natural commutative diagram with exact rows

$$\begin{array}{ccccc} (\tau_{\le i}F^\bullet)\otimes^L S & \longrightarrow & (\tau_{\le i+1}F^\bullet)\otimes^L S & \longrightarrow & (\mathcal{H}^i(F^\bullet))\otimes^L S \\ \downarrow & & \downarrow & & \downarrow \\ \tau_{\le i}(F^\bullet \otimes^L S) & \longrightarrow & \tau_{\le i+1}(F^\bullet \otimes^L S) & \longrightarrow & \mathcal{H}^i(F^\bullet \otimes^L S) \end{array}$$

Since $\mathcal{H}^i(F^\bullet)$ is flat, the right vertical map in an isomorphism. By induction assumption the middle vertical map is an isomorphism. This implies the statement.

(2) We start observing that by assumption and point (1) the natural map

$$E_2^{p,q}\otimes k = \mathrm{H}^a(\mathcal{H}^b(\mathcal{F}^\bullet))\otimes k \simeq \mathrm{H}^a(\mathcal{H}^b(\mathcal{F}^\bullet \otimes^L k)) = {}_kE_2^{p,q} \tag{3.1.7}$$

is an isomorphism. Let F_n^i be i^{th} piece of the filtration on $\mathrm{H}^n(\mathcal{F}^\bullet)$ induced by the spectral sequence. We prove by induction on $i \ge 0$ that, for every n, $E_2^{i,n} = E_\infty^{i,n}$ and F^{i+1} is locally free .

If $i = 0$, the equality $E_2^{0,n} = E_\infty^{0,n}$ follows from the fact that the edge map

$$\mathrm{H}^n(\mathcal{F}^\bullet) \to \mathrm{H}^0(\mathcal{H}^n(\mathcal{F}^\bullet))$$

is surjective, by (3.1.7), Nakayama's Lemma and the degeneration of the spectral sequence ${}_kE_2^{a,b}$. Then there is an exact sequence

$$0 \to F_n^1 \to \mathrm{H}^n(\mathcal{F}^\bullet) \to E_2^{0,n} \to 0$$

so that F_n^1 is finite locally free since $E_2^{0,n}$ and $\mathrm{H}^n(\mathcal{F}^\bullet)$ are.

Assume now $i > 0$. By induction hypothesis $E_2^{j,n} = E_\infty^{j,n-j}$ for $j < i$ and every n. In particular, all the morphisms with target $E_a^{i,n-i}$ have to vanish for every a, so that $E_\infty^{i,n-i} \subseteq E_2^{j,n-j}$ and there is a map $F^i \to E_2^{j,n-j}$. Again by Nakayama's Lemma, the degeneration of the spectral sequence ${}_kE_2^{a,b}$ and (3.1.7), this map is surjective. Hence $E_2^{j,n-j} = E_\infty^{j,n-j}$ and there is a short exact sequence

$$0 \to F_n^{i+1} \to F_n^i \to E_2^{i,n-i} \to 0,$$

so that F_n^{i+1} is finite and locally free, since $E_2^{i,n-i}$ and F^i are. This concludes the proof.

□

3.2. **Cohomological computations.** In this section we do the main computations on prismatic cohomology that we will need in the rest of the paper. Assumption ♥ gives a way to compute the Hodge groups, which is then used to study prismatic cohomology and the cohomology of the Nygaard filtration.

3.2.1. *Hodge cohomology groups.*

Lemma 3.2.1. *Assume that ♥ holds. Let R be a $\mathcal{O}_{\mathbb{C}_p}$-algebra. Then*

(1) The natural map

$$\mathrm{H}^i(\mathcal{X}, \Omega^j_{\mathcal{X}/\mathcal{O}_{\mathbb{C}_p}}) \otimes R \to \mathrm{H}^i(\mathcal{X}_R, \Omega^j_{\mathcal{X}_R/R})$$

is an isomorphism and $\mathrm{H}^i(\mathcal{X}_R, \Omega^j_{\mathcal{X}_R/R})$ is a free R-module of rank $h^{j,i}$.

(2) The Hodge to de-Rham spectral sequence

$$E_1^{a,b} := \mathrm{H}^b(\mathcal{X}_R, \Omega^a_{\mathcal{X}_R/R}) \Rightarrow \mathrm{H}^{a+b}_{\mathrm{dR}}(\mathcal{X}_R/R)$$

degenerates at the first page.

(3) The natural map

$$\mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_{\mathbb{C}_p}) \otimes R \to \mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}_R/R)$$

is an isomorphism and $\mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}_R/R)$ is a free R-module of rank h^n.

(4) If R is a complete coherent local ring of characteristic p with residue field k, then the conjugate spectral sequence

$$E_2^{a,b} := \mathrm{H}^a(\mathcal{X}_R, \mathcal{H}^b(\Omega^\bullet_{\mathcal{X}_R/R})) \Rightarrow \mathrm{H}^{a+b}_{\mathrm{dR}}(\mathcal{X}_R/R)$$

degenerates at the second page and $\mathrm{H}^a(\mathcal{X}_R, \tau_{\leq n}\Omega^\bullet_{\mathcal{X}_R/R})$ is a locally free R-module of rank $\sum_{i=0}^n h^{i,a-i}$.

Proof.

(1) Since $\mathcal{O}_{\mathbb{C}_p}$ is reduced, this follows from Grauert's theorem (see e.g. [Vak25, 25.1.5, p. 731] and [Vak25, Exercice 25.2.C (c), p. 740] for the extension to the non Noetherian situation), thanks to the assumption ♥.

(2) By point (1) and the functoriality of the Hodge to de-Rham spectral sequence, it is enough to show this for $R = \mathcal{O}_{\mathbb{C}_p}$. In this case, it follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^b(\mathcal{X}, \Omega^a_{\mathcal{X}/\mathcal{O}_{\mathbb{C}_p}}) & \xrightarrow{d} & \mathrm{H}^b(\mathcal{X}, \Omega^{a+1}_{\mathcal{X}/\mathcal{O}_{\mathbb{C}_p}}) \\ \downarrow & & \downarrow \\ \mathrm{H}^b(X, \Omega^a_{X/\mathbb{C}_p}) & \xrightarrow{d} & \mathrm{H}^b(X, \Omega^{a+1}_{X/\mathbb{C}_p}), \end{array}$$

since the spectral sequence always degenerates in characteristic 0 and the vertical maps are injective by point (1).

(3) Since the Hodge-to-de Rham spectral sequence degenerates at the first page by the previous point, the induced filtration F^i on $H^n_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_{\mathbb{C}_p})$ satisfies

$$F^i/F^{i+1} \simeq \mathrm{H}^{n-i}(\mathcal{X}, \Omega^i_{\mathcal{X}/\mathcal{O}_{\mathbb{C}_p}}),$$

in particular F^i/F^{i+1} is locally free. Hence one has a morphism of spectral sequences

$$\begin{array}{ccc} \mathrm{H}^b(\mathcal{X}, \Omega^a_{\mathcal{X}/\mathcal{O}_{\mathbb{C}_p}}) \otimes R & \Longrightarrow & \mathrm{H}^{a+b}_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_{\mathbb{C}_p}) \otimes R \\ \downarrow & & \downarrow \\ \mathrm{H}^b(\mathcal{X}_R, \Omega^a_{\mathcal{X}_R/R}) & \Longrightarrow & \mathrm{H}^{a+b}_{\mathrm{dR}}(\mathcal{X}_R/R) \end{array}$$

and the conclusion follows from point (1).

(4) By Cartier, $F_{X/R,*}\mathcal{H}^i(\Omega^\bullet_{\mathcal{X}_R/R}) \simeq \Omega^i_{\mathcal{X}_R^{(1)}/R}$, where $X^{(1)}$ is the Frobenius twist of X over R and $F_{X/R} : X \to X^{(1)}$ is the relative Frobenius. In particular

$$\mathcal{H}^i(\Omega^\bullet_{\mathcal{X}_R/R}) \quad \text{and} \quad \mathrm{H}^i(\mathcal{X}_R, \mathcal{H}^i(\Omega^\bullet_{\mathcal{X}_R/R}))$$

are R-flat, the second by point (1). Hence, by Lemma 3.1.6, it remains to show that

$$\mathrm{H}^a(Y, \mathcal{H}^b(\Omega^\bullet_{Y/k})) \Rightarrow \mathrm{H}^{a+b}_{\mathrm{dR}}(Y/k)$$

degenerates at the second page. This follows again from Cartier's isomorphism and point (2), since Y is proper over k.

□

3.2.2. *Prismatic cohomology groups.*

Proposition 3.2.2. *Assume that ♥ holds. Then:*

(1) $\mathrm{H}^n(\Delta^{(1)}/p)$ *and* $\mathrm{H}^n(\Delta/p)$ *are free* $\mathcal{O}_{\mathbb{C}_p^\flat}$*-modules of rank equal to* h^n.

(2) The natural map

$$\mathrm{H}^n(\Delta^{(1)}/p)) \otimes k \to \mathrm{H}^n_{\mathrm{dR}}(Y/k)$$

is an isomorphism.

(3) The natural maps

$$\mathrm{H}^n(\Delta^{(1)}/p)/d \to \mathrm{H}^n(\overline{\Delta}^{(1)}/p) \quad \textit{and} \quad \mathrm{H}^n(\Delta/p)/d \to \mathrm{H}^n(\overline{\Delta}/p)$$

are isomorphisms. In particular, $\mathrm{H}^n(\overline{\Delta}^{(1)}/p)$ *and* $\mathrm{H}^n(\overline{\Delta}/p)$ *are free* $\mathcal{O}_{\mathbb{C}_p^\flat}/d$*-modules of rank equal to* h^n*.*

(4) *The conjugate spectral sequence for* $\overline{\Delta}/p$

$$E_2^{a,b} := \mathrm{H}^a(\mathcal{H}^b(\overline{\Delta}/p)) \Rightarrow \mathrm{H}^{a+b}(\overline{\Delta}/p)$$

degenerates at the second page.

Proof.

(1) Since the Frobenius $\mathcal{O}_{\mathbb{C}_p^\flat} \to \mathcal{O}_{\mathbb{C}_p^\flat}$ is an isomorphism, by [BS22, Corollary 4.12] one has that $\mathrm{H}^n(\Delta^{(1)}/p) \simeq \mathrm{H}^n(\Delta/p) \otimes_{\varphi_{A_{\mathrm{inf}}}} \mathcal{O}_{\mathbb{C}_p^\flat}$, so that it is enough to prove the statement for $\Delta^{(1)}$. Since $\mathcal{O}_{\mathbb{C}_p^\flat}$ is a valuation ring with maximal ideal (d^{1/p^∞}), it is enough to show that the dimension of $\mathrm{H}^n(\Delta^{(1)}/p)[1/d]$ as $\mathbb{C}_p^\flat$-vector space is bigger or equal to the minimal number of generators of $\mathrm{H}^n(\Delta^{(1)}/p)/d$. By the étale comparison [BS22, Theorem 1.8], one has $\mathrm{H}^n(\Delta^{(1)}/p)[1/d] \simeq \mathrm{H}^n_{\text{ét}}(X, \mathbb{Z}/p) \otimes \mathbb{C}_p^\flat$, see for example [FKW24, Lemmas 2.1.6 and 2.2.10]. Hence it is enough to show that the minimal number of generators of $\mathrm{H}^n(\Delta^{(1)}/p)/d$ is smaller or equal to $\dim_{\mathbb{F}_p}(\mathrm{H}^n_{\text{ét}}(X, \mathbb{Z}/p))$. By (3.1.5), there is an isomorphism $\mathrm{H}^n(\Delta^{(1)}/p \otimes^L \mathcal{O}_{\mathbb{C}_p^\flat}/d) \simeq \mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}/p/\mathcal{O}_{\mathbb{C}_p^\flat}/d)$, hence an inclusion

$$\mathrm{H}^n(\Delta^{(1)}/p)/d \hookrightarrow \mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}/p/\mathcal{O}_{\mathbb{C}_p^\flat}/d).$$

By Lemma 3.2.1(3), it remains to show that $\dim_{\mathbb{C}_p}(\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C}_p) \leq \dim_{\mathbb{F}_p}(\mathrm{H}^n_{\text{ét}}(X, \mathbb{Z}/p))$, which follows from the fact that $\dim_{\mathbb{C}_p}(\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C}_p)) = \dim_{\mathbb{Q}_p}(\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_p))$ and the universal coefficients short exact sequence

$$0 \to \mathrm{H}^n_{\text{ét}}(X, \mathbb{Z}_p)/p \to \mathrm{H}^n_{\text{ét}}(X, \mathbb{Z}/p) \to \mathrm{H}^{n+1}_{\text{ét}}(X, \mathbb{Z}_p)[p] \to 0.$$

(2) This follows from (1) and Theorem 3.1.2.

(3) This follows from (1) and the universal coefficient theorem.

(4) Since $\mathcal{X}/p$ is smooth over $\mathcal{O}_{\mathbb{C}_p}/p$, the sheaf $\Omega^i_{\mathcal{X}/p/\mathcal{O}_{\mathbb{C}_p}/p}$ is finite locally free over $\mathcal{O}_{\mathbb{C}_p}/p$. By Lemma 3.2.1 and the previous point (2), $\mathrm{H}^i(\mathcal{X}/p, \Omega^j_{\mathcal{X}/p/\mathcal{O}_{\mathbb{C}_p}/p})$ and $\mathrm{H}^i(\overline{\Delta})$ are finite locally free $\mathcal{O}_{\mathbb{C}_p^\flat}/d$-modules. Hence Lemmas 3.1.6(2) and Theorem 3.1.2, imply that is is enough to show that the conjugate spectral sequence

$$E_2^{a,b} := \mathrm{H}^a(\mathcal{H}^b(\overline{\Delta} \otimes^L k)) \Rightarrow \mathrm{H}^{a+b}(\overline{\Delta} \otimes^L k)$$

for $\overline{\Delta} \otimes^L k$ degenerates at the second page. Since these are finitely dimensional k-vector spaces, it is enough to show that the conjugate spectral sequence

$$E_2^{a,b} := \mathrm{H}^a(\mathcal{H}^b(\varphi^*(\overline{\Delta} \otimes^L k))) \Rightarrow \mathrm{H}^{a+b}(\varphi^*(\overline{\Delta} \otimes^L k))$$

degenerates at the second page. By Theorem 3.1.2 and Lemma 3.2.1 the latter identifies with the conjugate spectral sequence for de-Rham cohomology of Y, which degenerates, again by Lemma 3.2.1.

□

3.2.3. *Nygaard filtration cohomology groups.*

Proposition 3.2.3. *Assume that ♥ holds. Then, the natural maps* $\mathrm{H}^n(N^{\geq i}/p) \xrightarrow{\iota} \mathrm{H}^n(\Delta^{(1)}/p)$ *are injective with image* $\phi^{-1}(d^i\mathrm{H}^n(\Delta/p))$*. In particular,* $\mathrm{H}^n(N^{\geq i}/p)$ *is a free* $\mathcal{O}_{\mathbb{C}_p^\flat}$*-module of rank equal to* h^n*.*

Proof. The proof is by induction on i, where the case $i = 0$ follows from the fact that $N^0 = \Delta^{(1)}$. Assume that $i > 0$ and that the statement holds for $j < i + 1$. Since

$$\mathrm{Im}(\mathrm{H}^n(N^{\geq i+1}/p) \xrightarrow{\iota} \mathrm{H}^n(\Delta^{(1)}/p)) \subseteq \phi^{-1}(d^{i+1}\mathrm{H}^n(\Delta^{(1)}/p)),$$

there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \mathrm{H}^n(N^{\geq i+1}/p) & \xrightarrow{\iota} & \mathrm{H}^n(\Delta^{(1)}/p) & \longrightarrow & \mathrm{H}^n((\Delta^{(1)}/p)/(N^{\geq i+1}/p)) & & \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle g} & & \\
0 & \to & \phi^{-1}(d^{i+1}\mathrm{H}^n(\Delta/p)) & \to & \mathrm{H}^n(\Delta^{(1)}/p) & \to & \mathrm{H}^n(\Delta^{(1)}/p)/\phi^{-1}(d^{i+1}\mathrm{H}^n(\Delta/p)) & \to & 0,
\end{array}$$

in which g is surjective. Hence the statement (and the inductive hypothesis) is equivalent to the injectivity of $g : \mathrm{H}^n((\Delta^{(1)}/p)/(N^{\geq i+1}/p)) \to \mathrm{H}^n(\Delta^{(1)}/p))/\phi^{-1}(d^{i+1}H^n(\Delta/p))$ for every n. Observe that $\phi : \mathrm{H}^n(\Delta^{(1)}/p) \to \mathrm{H}^n(\Delta/p)$ induces a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^n((\Delta^{(1)}/p)/(N^{\geq i+1}/p)) & \xrightarrow{\quad g \quad} & \mathrm{H}^n(\Delta^{(1)}/p)/\phi^{-1}(d^{i+1}\mathrm{H}^n(\Delta/p)) \\
 & {\scriptstyle\phi}\searrow \qquad \swarrow{\scriptstyle\phi} & \\
 & \mathrm{H}^n(\Delta/p)/d^{i+1}, &
\end{array}$$

where the map $\mathrm{H}^n(\Delta^{(1)}/p)/\phi^{-1}(d^{i+1}\mathrm{H}^n(\Delta^{(1)}/p)) \to \mathrm{H}^n(\Delta/p)/d^{i+1}$ is injective. Therefore, the statement (and the inductive hypothesis) is equivalent to the injectivity of ϕ : $\mathrm{H}^n((\Delta^{(1)}/p)/(N^{\geq i+1}/p)) \to \mathrm{H}^n(\Delta/p)/d^{i+1}$ for every n.

Consider the commutative diagram with exact rows

$$\text{(3.2.4)} \quad \begin{array}{ccccccccc}
 & & \mathrm{H}^n((N^{\geq i}/p)/(N^{\geq i+1}/p)) & \to & \mathrm{H}^n((\Delta^{(1)}/p)/(N^{\geq i+1}/p)) & \to & \mathrm{H}^n((\Delta^{(1)}/p)/(N^{\geq i}/p)) & & \\
 & & \downarrow{\scriptstyle\phi_i} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \\
0 & \to & \mathrm{H}^n(\Delta/p)/d \simeq \mathrm{H}^n(\overline{\Delta}/p) & \xrightarrow{d^i} & \mathrm{H}^n(\Delta/p)/d^{i+1} & \longrightarrow & \mathrm{H}^n(\Delta/p)/d^i & \longrightarrow & 0.
\end{array}$$

where the isomorphism in the first term of the bottom row follows from Proposition 3.2.2. By the inductive assumption, the right arrow is injective, hence it is enough to show that the map $\mathrm{H}^n((N^{\geq i}/p)/(N^{\geq i+1}/p)) \to \mathrm{H}^n(\overline{\Delta}/p)$ in injective. This is induced by the map of sheaves

$$\phi_i : (N^{\geq i}/p)/(N^{\geq i+1}/p) \to \overline{\Delta}/p,$$

which, by 3.1.2, induces an isomorphism $(N^{\geq i}/p)/(N^{\geq i+1}/p) \to \tau_{\leq i}\overline{\Delta}/p$. Hence it is enough to show that the natural map

$$\mathrm{H}^n(\tau_{\leq i}\overline{\Delta}/p) \to \mathrm{H}^n(\overline{\Delta}/p)$$

is injective, which follows again from Proposition 3.2.2. □

3.3. **Prismatic interpretation of p-adic vanishing cycles.**

3.3.1. *Vanishing cycles and d^i-fixed points of Frobenius.* The following is essentially [BMS19, Theorem 10.1], as explained in [Mor18, Remark 3.2]. It gives a relation between the action of Frobenius in prismatic cohomology, the Nygaard filtration and the sheaves of vanishing cycles.

Theorem 3.3.1. *If $j : X \to \mathcal{X}$ is the natural inclusion, then for every $i \geq 0$ then there is a natural long exact sequence*

$$\cdots \to \mathrm{H}^n(\mathcal{X}, \tau_{\leq i} Rj_*\mathbb{Z}/p) \to \mathrm{H}^n(N^{\geq i}/p) \xrightarrow{\iota-\varphi_i^{(1)}} \mathrm{H}^n(\Delta^{(1)}/p) \to \mathrm{H}^{n+1}(\tau_{\leq i} Rj_*\mathbb{Z}/p) \to \dots$$

Proof. This follows combining [BMS19, Theorem 10.1, Remark 10.3] and [BS22, Theorem 17.2]. More precisely, since $\mathcal{X}$ is proper, for every torsion étale complex of sheaves $\mathcal{F}$ over $\mathcal{X}$ the natural maps

$$\mathrm{H}^n(\mathcal{X}, \mathcal{F}) \to \mathrm{H}^n(\widehat{\mathcal{X}}, i_*\mathcal{F}) \to \mathrm{H}^n(Y, i^*\mathcal{F})$$

are isomorphisms, where we write $i : Y \to \mathcal{X}$ and $i : \widehat{\mathcal{X}} \to \mathcal{X}$ for the natural morphism. Hence, by [Hub96, Theorem 3.5.13, p. 207], there is a natural isomorphism

$$\mathrm{H}^n(\mathcal{X}, \tau_{\leq i} Rj_*\mathbb{Z}/p) \simeq \mathrm{H}^n(\widehat{\mathcal{X}}, \tau_{\leq i} Rb_*\mathbb{Z}/p),$$

where $b : \mathrm{Sh}_{\text{ét}}(\widehat{\mathcal{X}}_\eta) \to \mathrm{Sh}_{\text{ét}}(\widehat{\mathcal{X}})$ is the functor constructed in [Hub96, (3.5.12), p. 207] and $\widehat{\mathcal{X}}_\eta$ is the rigid generic fibre of $\widehat{\mathcal{X}}$. Then the conclusion follows from the exact triangle ([BMS18, Theorem 10.1])

$$\tau_{\leq i} Rb_*\mathbb{Z}/p \to N^{\geq i}/p \xrightarrow{\iota-\varphi_i^{(1)}} \Delta^{(1)}/p$$

and [BS22, Theorem 17.2]. □

We now explain when the long exact sequence in Theorem 3.3.1 can be broken in smaller pieces.

Lemma 3.3.2. *Assume that $n \leq i + 1$. Then the natural map*

$$\mathrm{H}^n(\tau_{\leq i} Rj_*\mathbb{Z}/p) \to \mathrm{H}^n(N^{\geq i}/p)$$

is injective.

Proof. By Theorem 3.3.1, it is enough to show that

$$\iota - \varphi_i^{(1)} : \mathrm{H}^{n-1}(N^{\geq i}/p) \to \mathrm{H}^{n-1}(\Delta^{(1)})$$

is surjective. By [LL23, Lemma 5.32], it is enough to show that $\mathrm{Coker}(\iota - \varphi_i^{(1)})$ is finite. Again by Theorem 3.3.1, it is enough to show that $\mathrm{H}^n(\tau_{\leq i} Rj_*\mathbb{Z}/p)$ is finite. Since $\tau_{>i} Rj_*\mathbb{Z}/p$ is concentrated in degrees $\geq i+1$ one has $\mathrm{H}^k(\tau_{>i} Rj_*\mathbb{Z}/p) = 0$ for $k \leq i$. Hence, since $n \leq i + 1$, the exact triangle

$$\tau_{\leq i} Rj_*\mathbb{Z}/p \to Rj_*\mathbb{Z}/p \to \tau_{>i} Rj_*\mathbb{Z}/p$$

shows that the natural map

$$\mathrm{H}^n(\tau_{\leq i} Rj_*\mathbb{Z}/p) \to \mathrm{H}^n(Rj_*\mathbb{Z}/p)$$

is injective (and even an isomorphism if $n \leq i$). This concludes the proof since

$$\mathrm{H}^n(Rj_*\mathbb{Z}/p) \simeq \mathrm{H}^n_{\text{ét}}(X, \mathbb{Z}/p)$$

is a finite dimensional $\mathbb{F}_p$-vector space. □

Observe that the natural map

$$s : \mathrm{H}^n(\Delta/p) \to \mathrm{H}^n(\Delta^{(1)}/p)$$

is an isomorphism, hence we can build the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^n(\tau_{\leq i} Rj_*\mathbb{Z}/p) & \longrightarrow & \mathrm{H}^n(N^{\geq i}/p) & \xrightarrow{\iota-\varphi_i^{(1)}} & \mathrm{H}^n(\Delta^{(1)}/p) \\ & & \downarrow & & \downarrow{\scriptstyle s^{-1}\circ\iota} & & \downarrow{\scriptstyle \cdot d^i\circ s^{-1}} \\ 0 & \longrightarrow & \mathrm{Ker}(\varphi - d^i) & \longrightarrow & \mathrm{H}^n(\Delta/p) & \xrightarrow{d^i-\varphi} & \mathrm{H}^n(\Delta/p), \end{array} \tag{3.3.3}$$

where the left vertical map is induced by the other two vertical ones.

Lemma 3.3.4. *Assume that ♥ holds. Then, for $n \leq i+1$, the natural map*

$$\mathrm{H}^n(\tau_{\leq i} Rj_*\mathbb{Z}/p) \to \mathrm{Ker}(\varphi - d^i)$$

is an isomorphism.

Proof. By Lemma 3.3.2 and Theorem 3.3.1, the rows of the diagram (3.3.3) are exact. By Propositions 3.2.2 and 3.2.3, the two right vertical arrows are injective. So the conclusion follows from the fact that

$$\mathrm{H}^n(N^{\geq i}/p) = \phi^{-1}(d^i \mathrm{H}^n(\Delta/p)),$$

again by Proposition 3.2.3. □

3.3.2. *Proof of Theorem 1.3.2.* By construction, the edge map

$$\mathrm{H}^n(X_{\mathbb{C}_p}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, R^n j_*\mathbb{Z}/p),$$

in the Leray spectral sequence is induced by the exact triangle

$$\tau_{\leq n-1}(Rj_*\mathbb{Z}/p) \to Rj_*\mathbb{Z}/p \to \tau_{>n-1}(Rj_*\mathbb{Z}/p),$$

observing that $\mathrm{H}^n(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \tau_{>n-1}(Rj_*\mathbb{Z}/p)) \subseteq \mathrm{H}^0(X_{\mathcal{O}_{\mathbb{C}_p}}, R^n j_*\mathbb{Z}/p)$. Hence

$$\mathrm{Ker}(\mathrm{H}^n(X_{\mathbb{C}_p}, \mathbb{Z}/p) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, R^n j_*\mathbb{Z}/p)) = \mathrm{H}^n(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \tau_{\leq n-1}(Rj_*\mathbb{Z}/p)),$$

and the conclusion follows from Lemma 3.3.4.

4. Total Newton slope and Hodge polygons

In this section we prove Theorem 1.4.1. We start in Section 4.1 by defining and studying the total Newton slope and the Hodge polygon for abstract finite free modules over $\mathcal{O}_{\mathbb{C}_p^\flat}$ endowed with a Frobenius, taking inspiration from [Kat79]. Almost by construction, the total Newton slope will be at least the height of the Hodge polygon; in fact, in Lemma 4.1.11 we show that they are equal. Then in Section 4.2 we specialise to the geometric setting and control this height via the Hodge numbers of X, proving Proposition 4.2.2, which in turn implies Theorem 1.4.1.

4.1. **Total Newton slope and Hodge polygon.** For this section, let M be a finite free $\mathcal{O}_{\mathbb{C}_p^\flat}$-module of rank $r \in \mathbb{Z}_{\geq 1}$ equipped with a Frobenius-semi-linear map φ that becomes an isomorphism after inverting d. If M and N are such modules, we write $\mathrm{Hom}_\varphi(M, N)$ for the $\mathbb{F}_p$ vector space of morphisms as $\mathcal{O}_{\mathbb{C}_p^\flat}$-modules that commute with the Frobenius.

Recall that, by [Mil16, Lemma 4.13, p. 128], for every integer i,

$$M^{\varphi=d^i} := \mathrm{Ker}(\varphi - d^i : M \to M)$$

is an $\mathbb{F}_p$-vector space of rank at most r. If $\mathcal{B} := e_1, \ldots, e_r$ is a basis of M, we write $M_\mathcal{B}(\varphi) \in M_r(\mathcal{O}_{\mathbb{C}_p^\flat})$ for the matrix whose i^{th}-column is the coordinates vector of $\varphi(e_i)$ with respect to $\mathcal{B}$. Furthermore, we chose a compatible system $\{d^{1/r}\}_{r\in\mathbb{N}}$ of roots of d.

4.1.1. *Total Newton slope.*

Definition 4.1.1. Given a rational number $\alpha \geq 0$, we denote by $\mathcal{O}(-\alpha)$ the rank one $\mathcal{O}_{\mathbb{C}_p^\flat}$-module

$$\mathcal{O}(-\alpha) = \mathcal{O}_{\mathbb{C}_p^\flat} \cdot e$$

with Frobenius φ given by $\varphi(e) = d^\alpha e$.

By construction, one has $\mathcal{O}(-\alpha)\otimes\mathcal{O}(-\beta) \simeq \mathcal{O}(-\alpha-\beta)$. These $\mathcal{O}(-\alpha)$ are all the possible rank 1 modules with Frobenius, as the following lemma shows.

Lemma 4.1.2. *If $r = 1$, then $M \simeq \mathcal{O}(-\alpha)$ for a unique $\alpha \in \mathbb{Q}_{\geq 0}$.*

Proof. Let $x \in M$ be a generator and write $\varphi(x) = \lambda x$. Since $\mathcal{O}_{\mathbb{C}_p^\flat}$ is a valuation ring whose maximal ideal is generated by d^α for $\alpha \in \mathbb{Q}_{>0}$, there exist $\alpha \in \mathbb{Q}_{\geq 0}$ and $\mu \in \mathcal{O}^*_{\mathbb{C}_p^\flat}$ such that $\lambda = d^\alpha \mu$. Then $x := e/\mu^{\frac{1}{p-1}}$ is a generator of M such that $\varphi(x) = d^\alpha x$, so that $M \simeq \mathcal{O}(-\alpha)$. For uniqueness, it is enough to observe that $\mathrm{Hom}_\varphi(\mathcal{O}(-\alpha), \mathcal{O}(-\beta)) \neq 0$ if and only if $\alpha \geq \beta$. □

Proposition 4.1.3.

(1) *There exists a Frobenius-stable decreasing filtration $F^\bullet$ of M, such that*

$$\mathrm{Gr}_i(F^\bullet) := F^i/F^{i+1} \simeq \mathcal{O}(-\alpha_i)$$

for some $\alpha_i \in \mathbb{Q}_{\geq 0}$.

(2) *If $F^\bullet$ and $G^\bullet$ are two Frobenius-stable decreasing filtrations such that*

$$\mathrm{Gr}_i(F^\bullet) \simeq \mathcal{O}(-\alpha_i) \quad \textit{and} \quad \mathrm{Gr}_i(G^\bullet) \simeq \mathcal{O}(-\beta_i)$$

for some $\alpha_i, \beta_i \in \mathbb{Q}_{\geq 0}$, then

$$\sum_i \alpha_i = \sum_i \beta_i.$$

Proof.

(1) We prove this by induction on r. The case $r = 1$ is Lemma 4.1.2, so assume $r > 1$. By the following Lemma 4.1.4, the set

$$\mathcal{E}(M, \varphi) := \{x \in M \mid \exists\, i \in \mathbb{Q}_{\geq 0} \text{ such that } \varphi(x) = d^i x \text{ and } x \notin \mathfrak{m}^\flat M\}$$

is finite and non-empty. In particular, there exists a minimal $i \in \mathbb{Q}_{\geq 0}$ such that there exists an $x \in M$ with $\varphi(x) = d^i x$. Fix such an x; then the $\mathcal{O}_{\mathbb{C}_p^\flat}$-module $\langle x\rangle$ generated by x is isomorphic to $\mathcal{O}(-i)$. Thus, by induction it is enough to show that $M/\langle x\rangle$ is torsion-free (hence free, since $\mathcal{O}_{\mathbb{C}_p^\flat}$ is a valuation ring). It is enough to show that if $y \in M$ and $j \in \mathbb{Q}_{\geq 0}$ are such that $d^j y = \lambda x$ for some $\lambda \in \mathcal{O}_{\mathbb{C}_p^\flat}$, then $a := v_d(\lambda) \geq j$. Upon rescaling j, we can assume that $y \in M \setminus \mathfrak{m}^\flat M$ and upon multiplying it by a unit, we can assume that $\lambda = d^a$. Then we have

$$d^{pj}\varphi(y) = \varphi(d^j y) = \varphi(\lambda x) = \lambda^p d^i x = \lambda^{p-1} d^{i+j} y = d^{a(p-1)+i+j} y,$$

hence $\varphi(y) = d^{a(p-1)+i-j(p-1)} y$. By the minimality of i and since $y \in M \setminus \mathfrak{m}^\flat M$ we get $a(p-1) + i - j(p-1) \geq i$, hence $a \geq j$.

(2) Observe that

$$\mathcal{O}\left(-\sum_{i=1}^r \alpha_i\right) \simeq \otimes_{i=1}^r \mathrm{Gr}^i(F^\bullet) \simeq \wedge^r M \simeq \otimes_{i=1}^r \mathrm{Gr}^i(G^\bullet) \simeq \mathcal{O}\left(-\sum_{i=1}^r \beta_i\right),$$

so the conclusion follows from Lemma 4.1.2. □

Lemma 4.1.4. *The set*

$$\mathcal{E}(M,\varphi) := \{x \in M \mid \exists\, i \in \mathbb{Q}_{\geq 0} \text{ such that } \varphi(x) = d^i x \text{ and } x \notin \mathfrak{m}^\flat M\}$$

is finite and non-empty.

Proof. Recall that by [Mil16, Lemma 4.13, p.128], $M\left[\frac{1}{d}\right]^{\varphi=1}$ is an $\mathbb{F}_p$-vector space of dimension r, in particular it is finite and non-empty. Hence, to prove the lemma it is enough to construct a bijection

$$f : \mathcal{E}(M,\varphi) \to M\left[\frac{1}{d}\right]^{\varphi=1}.$$

For $x \in \mathcal{E}(M,\varphi)$, we define $f(x) := \frac{x}{d^{i/(p-1)}}$, where $i \in \mathbb{Q}_{\geq 0}$ is the unique rational such that $\varphi(x) = d^i x$. Since the map is well defined and surjective by definition, we just need to show injectivity. If $x, y \in \mathcal{E}(M,\varphi)$ map to the same element, then

$$\frac{x}{d^{i/(p-1)}} = \frac{y}{d^{j/(p-1)}} \quad \text{hence} \quad d^{j/(p-1)}x = d^{i/(p-1)}y.$$

If $i = j$, we immediately get $x = y$. If $i \neq j$, then without loss of generality we can assume $j > i$ and hence

$$x = d^{(j-i)/(p-1)}y.$$

This implies $x \in \mathfrak{m}^\flat M$, which is a contradiction. □

Remark 4.1.5. The existence of a filtration as in Proposition 4.1.3 is equivalent to the existence of a basis $\mathcal{B}$ of M such that the matrix $M_{\mathcal{B}}(\varphi)$ is upper triangular with d^{α_i} on the diagonal. In particular, $\mathrm{TS}(M)$ is the d-adic valuation of the determinant of $M_{\mathcal{C}}(\varphi)$ for any basis $\mathcal{C}$ of M.

Definition 4.1.6. The total slope of M is

$$\mathrm{TS}(M) := \sum \alpha_i,$$

where the $\alpha_i \in \mathbb{Q}_{\geq 0}$ are the ones appearing in the graded quotients for any Frobenius-stable decreasing filtration $F^\bullet$ of M such that $\mathrm{Gr}_i(F^\bullet) \simeq \mathcal{O}(-\alpha_i)$ for some $\alpha_i \in \mathbb{Q}_{\geq 0}$. The total slope is well defined thanks to Proposition 4.1.3(2).

Example 4.1.7. By contrast, the set of "slopes", i.e. the set of α_i appearing in the filtration, is not well defined, contrary to what one could expect by analogy with the crystalline situation (see e.g. [Kat79]), as the following example shows. To simplify the computation, we assume that $p = 2$. Let $M = \mathcal{O}_{\mathbb{C}_p^\flat} \oplus \mathcal{O}_{\mathbb{C}_p^\flat}$ be endowed with a Frobenius whose matrix with respect to the standard basis $\mathcal{B}$ of M is given by

$$M_{\mathcal{B}}(\varphi) = \begin{bmatrix} d^2 & 1 \\ 0 & d^{1/2} \end{bmatrix}.$$

With respect to the filtration induced by $\mathcal{B}$, the "slopes" are $1/2$ and 2. On the other hand, by Hensel's lemma, there exists $x \in \mathcal{O}_{\mathbb{C}_p^\flat}$ such that $dx^2 + x + 1 = 0$ and such that the reduction of x modulo d is 1. In particular, x is a unit. Hence, the elements $(x, d^{1/2})$ and $(0, 1)$ form a new basis $\mathcal{C}$ of M. One computes that

$$M_{\mathcal{C}}(\varphi) = \begin{bmatrix} d & x^{-1} \\ 0 & xd^{3/2} \end{bmatrix},$$

so that with respect to the filtration induced by $\mathcal{C}$, the "slopes" are 1 and $3/2$. Hence the set of "slopes" is not well defined, as claimed. As an additional remark, observe that in the first filtration the “slopes” are increasing, while in the second one they are decreasing (and neither of the filtrations splits). Thus, even the order of the “slopes” is not well defined.

4.1.2. *Hodge polygon.* Since $M/\varphi(M)$ is finitely presented and $\mathcal{O}_{\mathbb{C}_p^\flat}$ is a valuation ring whose maximal ideal is generated by $(d^a)_{a\in\mathbb{Q}_{>0}}$, by [Sta25, Tag 0ASP], we have

$$\frac{M}{\varphi(M)} \simeq \bigoplus_i \frac{\mathcal{O}_{\mathbb{C}_p^\flat}}{d^{\beta_i}} \tag{4.1.8}$$

for unique $\beta_i \in \mathbb{Q}$ and $0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_r$. We call $\{\beta_i\}$ the Hodge slopes of M.

Remark 4.1.9. By [Sta25, Tag 0AST], there are two bases $\mathcal{B}$, $\mathcal{B}'$ of M such that the matrix of φ associated to $\mathcal{B}$, $\mathcal{B}'$ is diagonal with entries $\beta_1, \dots, \beta_r$. In particular, one has

$$\sum_{i=1}^r \beta_i = v_d(\det(M_{\mathcal{C}}(\varphi))),$$

for any basis $\mathcal{C}$ of M.

Definition 4.1.10. Let $P_0 := (0,0) \in \mathbb{R}^2$ and for $1 \le j \le r$, let

$$P_j := (j, \sum_{i=1}^j \beta_i) \in \mathbb{R}^2$$

We define the *Hodge polygon* of M as the union, for $0 \le j \le r-1$, of the segments that join P_j and P_{j+1}. The *height* $h(M)$ of the Hodge polygon of M is the height of the end point, that is, the sum of all β_i's.

Lemma 4.1.11. *The total slope of M is equal to the height of the Hodge polygon of M.*

Proof. This follows from the fact that both are the d-adic valuation of the determinant of $M_{\mathcal{B}}(\varphi)$ for any basis $\mathcal{B}$ of φ, see Remarks 4.1.5 and 4.1.9. □

4.2. **Hodge polygon and Hodge numbers.** Let $\mathcal{X}$ be a smooth proper scheme over $\mathcal{O}_{\mathbb{C}_p}$ and $\widehat{\mathcal{X}}$ its formal p-adic completion. We also write Y for $\mathcal{X}_k$ and X for $\mathcal{X}_{\mathbb{C}_p}$. We let

$$h^{a,b} := \dim_{\mathbb{C}_p}(\mathrm{H}^b(X, \Omega^a_{X/\mathbb{C}_p})) \quad \text{and} \quad h^n := \dim_{\mathbb{C}_p}(\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C}_p)) = \dim_{\mathbb{Q}_p}(\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_p))$$

If ♥ holds, then $\mathrm{H}^n(\Delta/p)$ is a finite free $\mathcal{O}_{\mathbb{C}_p^\flat}$-module of rank h^n by Proposition 3.2.2. Hence, $\mathrm{H}^n(\Delta/p)$ has a total Newton slope $\mathrm{TS}(\mathrm{H}^n(\Delta/p))$, a Hodge polygon and a Hodge height $h(\mathrm{H}^n(\Delta/p))$. We now explain how to compute them from the geometry of $\mathcal{X}$.

Definition 4.2.1. Let $Q_0 := (0,0) \in \mathbb{R}^2$ and for $1 \le j \le h^n$, let

$$Q_j := (\sum_{i=0}^j h^{i,n-i}, \sum_{i=0}^j i \cdot h^{i,n-i}) \in \mathbb{R}^2.$$

We define the *geometric Hodge polygon* of $\mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_{\mathbb{C}_p})$ as the union, for $0 \le j \le r-1$, of the segments that join Q_j and Q_{j+1}.

Proposition 4.2.2. *Assume that ♥ holds. Then the Hodge polygon of* $\mathrm{H}^n(\Delta/p)$ *coincides with the geometric Hodge polygon of* $\mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_{\mathbb{C}_p})$.

Combining Proposition 4.2.2 and Lemma 4.1.11, we get Theorem 1.4.1.

Proof of Proposition 4.2.2. By definition, we need to show that that there exists an isomorphism

$$\mathrm{H}^n(\Delta/p)/\varphi(\mathrm{H}^n(\Delta/p)) \simeq \bigoplus_{j=0}^{n} (\mathcal{O}_{\mathbb{C}_p^\flat}/d^j)^{h^{j,n-j}},$$

which in turn is equivalent to showing that there exists an isomorphism

$$\mathrm{H}^n(\Delta/p)/\phi(\mathrm{H}^n(\Delta^{(1)}/p)) \simeq \bigoplus_{j=0}^{n} (\mathcal{O}_{\mathbb{C}_p^\flat}/d^j)^{h^{j,n-j}}.$$

This is equivalent to showing that, for every $i \in \mathbb{Z}_{\geq 0}$, there exists an isomorphism

$$\mathrm{H}^n(\Delta/p)/(\phi(\mathrm{H}^n(\Delta^{(1)}/p)), d^i) \simeq \bigoplus_{j=0}^{i-1} (\mathcal{O}_{\mathbb{C}_p^\flat}/d^j)^{h^{j,n-j}} \oplus (\mathcal{O}_{\mathbb{C}_p^\flat}/d^i)^{\sum_{j\geq i} h^{j,n-j}}.$$

Choose bases of $\mathrm{H}^n(\Delta^{(1)}/p))$ and $\mathrm{H}^n(\Delta/p)$ so that the matrix of ϕ with respect to these bases is diagonal (see Remark 4.1.9) with entries the Hodge slopes $\{\beta_1 \dots \beta_r\}$ of $\mathrm{H}^n(\Delta/p)$. Then, the commutative diagram with exact rows and surjective vertical arrows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^n(\Delta^{(1)}/p)) & \xrightarrow{\phi} & \mathrm{H}^n(\Delta/p) & \longrightarrow & \frac{\mathrm{H}^n(\Delta/p)}{\phi(\mathrm{H}^n(\Delta^{(1)}/p))} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\mathrm{H}^n(\Delta^{(1)}/p))}{\phi^{-1}(d^i\mathrm{H}^n(\Delta/p))} & \xrightarrow{\phi} & \frac{\mathrm{H}^n(\Delta/p)}{d^i} & \longrightarrow & \frac{\mathrm{H}^n(\Delta/p)}{(\phi(\mathrm{H}^n(\Delta^{(1)}/p)),d^i)} & \longrightarrow & 0
\end{array}$$

induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{\mathrm{H}^n(\Delta^{(1)}/p)}{\phi^{-1}(d^i\mathrm{H}^n(\Delta/p))} & \xrightarrow{\phi} & \frac{\mathrm{H}^n(\Delta/p)}{d^i} & \longrightarrow & \frac{\mathrm{H}^n(\Delta/p)}{(\phi(\mathrm{H}^n(\Delta^{(1)}/p)),d^i)} & \longrightarrow & 0 \\
 & & \phi \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
0 & \longrightarrow & \bigoplus_{\beta_j<i} \frac{d^{\beta_j}\mathcal{O}_{\mathbb{C}_p^\flat}}{d^i} & \longrightarrow & \frac{\mathcal{O}^r_{\mathbb{C}_p^\flat}}{d^i} & \longrightarrow & \bigoplus_{\beta_j<i} \frac{\mathcal{O}_{\mathbb{C}_p^\flat}}{d^{\beta_j}} \oplus \bigoplus_{\beta_j\geq i} \frac{\mathcal{O}_{\mathbb{C}_p^\flat}}{d^i} & \longrightarrow & 0,
\end{array}$$

in which the vertical arrows are isomorphisms, the bottom left arrow is the natural inclusion and $r := h^n$. Hence, it is enough to show that there exists an isomorphism

$$\mathrm{H}^n(\Delta^{(1)}/p)/\phi^{-1}(d^i\mathrm{H}^n(\Delta/p)) \simeq \bigoplus_{j=0}^{i-1} (d^j\mathcal{O}_{\mathbb{C}_p^\flat}/d^i)^{h^{j,n-j}}.$$

We prove this by induction on i. The case $i = 0$ is trivial, since all the groups involved are zero, so assume $i \geq 1$. By Proposition 3.2.3 one has

$$\phi^{-1}(d^i\mathrm{H}^n(\Delta/p)) = \mathrm{H}^n(N^{\geq i}/p) \quad \text{and}$$

$$\mathrm{H}^n(\Delta^{(1)}/p)/\mathrm{H}^n(N^{\geq i}/p) \simeq \mathrm{H}^n((\Delta^{(1)}/p)/(N^{\geq i}/p)) \simeq \mathrm{H}^n(\Delta^{(1)}/p)/\phi^{-1}(d^i\mathrm{H}^n(\Delta/p)).$$

Thanks to Proposition 3.2.3, we have a commutative diagram with exact rows and injective vertical maps (cf. (3.2.4))

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^n((N^{\geq i-1}/p)/(N^{\geq i}/p)) & \to & \mathrm{H}^n(\Delta^{(1)}/p)/\mathrm{H}^n(N^{\geq i}/p) & \to & \mathrm{H}^n(\Delta^{(1)}/p)/\mathrm{H}^n(N^{\geq i-1}/p) & \to & 0 \\
 & & \downarrow{\scriptstyle \phi_{i-1}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & \mathrm{H}^n(\Delta/p)/d & \xrightarrow{d^{i-1}} & \mathrm{H}^n(\Delta/p)/d^i & \longrightarrow & \mathrm{H}^n(\Delta)/d^{i-1} & \longrightarrow & 0
\end{array}$$

which can be identified, via the choice of the basis and the inductive hypothesis, with

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{H}^n\left(\frac{N^{\geq i-1}/p}{N^{\geq i}/p}\right) & \to & \bigoplus_{\beta_j > i-1} \frac{d^{\beta_j}\mathcal{O}_{\mathbb{C}_p^\flat}}{d^i} \oplus \bigoplus_{j=0}^{i-2} \left(\frac{d^j\mathcal{O}_{\mathbb{C}_p^\flat}}{d^i}\right)^{h^{j,n-j}} & \to & \bigoplus_{j=0}^{i-2} \left(\frac{d^j\mathcal{O}_{\mathbb{C}_p^\flat}}{d^{i-1}}\right)^{h^{j,n-j}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathcal{O}_{\mathbb{C}_p^\flat}/d)^r & \xrightarrow{d^{i-1}} & (\mathcal{O}_{\mathbb{C}_p^\flat}/d^i)^r & \longrightarrow & (\mathcal{O}_{\mathbb{C}_p^\flat}/d^{i-1})^r & \longrightarrow & 0,
\end{array}$$

in which the upper right map is induced by the canonical projection $d^j\mathcal{O}_{\mathbb{C}_p^\flat}/d^i \to d^j\mathcal{O}_{\mathbb{C}_p^\flat}/d^{i-1}$. It remains to show that $\mathrm{H}^n((N^{\geq i-1}/p)/(N^{\geq i}/p))$ is a finite free $\mathcal{O}_{\mathbb{C}_p^\flat}/d$-module of rank $\sum_{j=0}^{i} h^{j,n-j}$. To prove this, observe that, by Theorem 3.1.2, one has an isomorphism

$$(N^{\geq i-1}/p)/(N^{\geq i}/p) \simeq \tau_{\leq i}(\overline{\Delta}/p).$$

On the other hand, recall from Theorem 3.1.2 that $\mathcal{H}^b(\overline{\Delta}/p) \simeq \Omega^b_{\mathfrak{X}/p/\mathcal{O}_{\mathbb{C}_p^\flat}/d}$, so that $\mathrm{H}^a(\mathcal{H}^b(\overline{\Delta}/p))$ is a free $\mathcal{O}_{\mathbb{C}_p^\flat}/d$-module of rank $h^{b,a}$, by Lemma 3.2.1. The conclusion now follows from the degeneration of the conjugate spectral sequence for $\overline{\Delta}/p$, see Proposition 3.2.2. □

Remark 4.2.3. It follows from Proposition 4.2.2 that the Hodge slopes β_i are integers.

5. Proof of the main result

In this section we prove Theorems 1.3.3 and 1.2.1, collecting the fruits of the work done in the previous sections. We start in Section 5.1 by proving a useful semi-linear algebra lemma. The proofs of Theorems 1.3.3 and 1.2.1 are then contained in Sections 5.2 and 5.3, respectively.

5.1. **A semi-linear algebra lemma.**

Lemma 5.1.1. *Let M be a finite free $\mathcal{O}_{\mathbb{C}_p^\flat}$-module of rank $r \in \mathbb{Z}_{\geq 1}$ equipped with a Frobenius semi-linear map $\varphi : M \to M$ that becomes an isomorphism after inverting d. Assume that* $\mathrm{TS}(M) = ir$. *Then*

$$dim_{\mathbb{F}_p}(M^{\varphi = d^i}) = r$$

if and only if

$$M \simeq \bigoplus_{i=1}^{r} \mathcal{O}(-i).$$

Proof. The only if implication being clear, we prove the if direction. To do this, we argue by induction on r, the case $r = 1$ being a direct computation. So assume $r > 1$ and that $\dim_{\mathbb{F}_p}(M^{\varphi=d^i}) = r$. By Proposition 4.1.3, there exists a Frobenius-stable decreasing filtration $F^\bullet$ of M and $a_1 \dots a_r \in \mathbb{Q}_{\geq 0}$ such that

$$Gr_j := F^j/F^{j+1} \simeq \mathcal{O}(-a_j) \quad \text{and} \quad \sum_{j=1}^{r} a_j = ir.$$

Since

$$r = \dim_{\mathbb{F}_p}(M^{\varphi=d^i}) \leq \sum_{j=1}^{r} \dim_{\mathbb{F}_p}(Gr_j^{\varphi=d^i}) \quad \text{and} \quad \mathcal{O}(-a_j)^{\varphi=d^i} \neq 0 \Leftrightarrow a_j \leq i,$$

the assumption $\dim_{\mathbb{F}_p}(M^{\varphi=d^i}) = r$ implies that $a_j \leq i$ for every j. Since $\sum_{j=1}^{r} a_j = ir$, this in turn implies that $a_j = i$ for every j. Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (F^1)^{\varphi=d^i} & \longrightarrow & M^{\varphi=d^i} & \longrightarrow & \mathcal{O}(-i)^{\varphi=d^i} = \mathbb{F}_p & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^1 & \longrightarrow & M & \longrightarrow & \mathcal{O}(-i) & \longrightarrow & 0. \\
 & & \downarrow{\scriptstyle d^i-\varphi} & & \downarrow{\scriptstyle d^i-\varphi} & & \downarrow{\scriptstyle d^i-\varphi} & & \\
0 & \longrightarrow & F^1 & \longrightarrow & M & \longrightarrow & \mathcal{O}(-i) & \longrightarrow & 0.
\end{array}$$

Since $\dim_{\mathbb{F}_p}((F^1)^{\varphi=d^i}) \leq \operatorname{rank}_{\mathcal{O}_{\mathbb{C}_p^\flat}}((F^1) = r-1$ and $\dim_{\mathbb{F}_p}(M^{\varphi=d^i}) = r$, the upper row shows that $\dim_{\mathbb{F}_p}((F^1)^{\varphi=d^i}) = r-1$. Since $\mathrm{TS}(F^1) = i(r-1)$, by induction $F^1 \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}(-i)$, hence it is enough to show that the surjective map $M \to \mathcal{O}(-i)$ admits a φ-equivariant section. Since $\dim_{\mathbb{F}_p}((F^1)^{\varphi=d^i}) = r-1$ and $\dim_{\mathbb{F}_p}(M^{\varphi=d^i}) = r$, the map $M^{\varphi=d^i} \to \mathcal{O}(-i)^{\varphi=d^i}$ is surjective. The $\mathcal{O}_{\mathbb{C}_p^\flat}$-module $\mathcal{O}(-i)$ is generated by an element e such that $\varphi(e) = d^i e$, thus there exists an element $x \in M^{\varphi-d^i} \subseteq M$ mapping to e and the map $\mathcal{O}(-i) \to M$ sending e to x gives a φ-equivariant splitting of the surjection $M \to \mathcal{O}(-i)$. □

Example 5.1.2. In Lemma 5.1.1, to conclude that $M \simeq \bigoplus_{i=1}^{r} \mathcal{O}(-i)$ it is not enough to assume that $\dim_{\mathbb{F}_p}(M^{\varphi=d^i}) = r$. Some assumptions on $\mathrm{TS}(M)$ are necessary, as the following example shows. Let $M = \mathcal{O}_{\mathbb{C}_p^\flat} \oplus \mathcal{O}_{\mathbb{C}_p^\flat}$ be endowed with the Frobenius φ whose matrix with respect to the standard basis $\mathcal{B}$ of M is given by

$$M_{\mathcal{B}}(\varphi) := \begin{bmatrix} 0 & d \\ 1 & 0 \end{bmatrix}$$

Then $\mathrm{TS}(M) = 1$, so that $M \not\simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, but

$$M^{\varphi-d} = \{(a^p d^{\frac{p}{p^2-1}}, a d^{\frac{1}{p^2-1}}) \in \mathcal{O}_{\mathbb{C}_p^\flat} \oplus \mathcal{O}_{\mathbb{C}_p^\flat} \text{ for } a \in \mathbb{F}_{p^2}\}$$

is a 2-dimensional $\mathbb{F}_p$-vector space.

5.2. **Proof of Theorem 1.3.3.** Arguing by contradiction, we assume that

$$\dim_{\mathbb{F}_p}(\mathrm{Ker}(\varphi - d^n : \mathrm{H}^{2n}(\Delta/p) \to \mathrm{H}^{2n}(\Delta/p)) = \mathrm{rank}_{\mathcal{O}_{\mathbb{C}_p^\flat}}(\mathrm{H}^{2n}(\Delta/p))$$

By Theorem 1.4.1, one has

$$\mathrm{TS}(\mathrm{H}^{2n}(\Delta/p)) = \sum_{j=0}^{2n} j \cdot h^{j,2n-j} = n \cdot h^{2n} = n \cdot \mathrm{rank}_{\mathcal{O}_{\mathbb{C}_p^\flat}}(\mathrm{H}^{2n}(\Delta/p)),$$

so we can apply Lemma 5.1.1 to deduce that

$$\mathrm{H}^{2n}(\Delta/p) \simeq \mathcal{O}(-n)^{\mathrm{rank}_{\mathcal{O}_{\mathbb{C}_p^\flat}}(\mathrm{H}^{2n}(\Delta/p))}.$$

Hence, the Hodge polygon of $\mathrm{H}^{2n}(\Delta/p)$ is a straight line with slope n. By Proposition 4.2.2, this implies the same for the geometric Hodge polygon of $H^{2n}_{\mathrm{dR}}(X/\mathbb{C}_p)$. By definition, this means that $\mathrm{H}^{2n}(X/\mathbb{C}_p) = \mathrm{H}^n(X, \Omega^n_{X/\mathbb{C}_p})$, that is, $\mathrm{H}^i(X, \Omega^{2n-i}_{X/\mathbb{C}_p}) = 0$ for $i \neq n$, which concludes the proof.

Example 5.2.1. Assume that $\mathcal{E}$ is an elliptic curve such that $\mathrm{H}^1(\mathcal{E}, \Delta/p) \simeq \mathcal{O}^2_{\mathcal{O}_{\mathbb{C}_p}}$ with Frobenius given, as in Example 5.1.2, by the matrix

$$M_{\mathcal{B}}(\varphi) = \begin{bmatrix} 0 & d \\ 1 & 0, \end{bmatrix}$$

so that there is an exact sequence

$$0 \to \mathcal{O}\Big(\frac{-p}{p+1}\Big) \to \mathrm{H}^1(\mathcal{E}, \Delta/p) \to \mathcal{O}\Big(\frac{-1}{p+1}\Big) \to 0.$$

Let $\mathcal{X} = \mathcal{E}^3$, so that $\mathrm{H}^3(\Delta/p) = \wedge^3((\mathrm{H}^1(\mathcal{E}, \Delta/p))^3)$. Hence the maximal "slope" appearing in the filtration of $\mathrm{H}^3(\Delta/p)$ is $(3p/p+1)$, which is greater than 2 if and only if $p > 2$. Hence to guarantee that

$$\dim_{\mathbb{F}_p}(\mathrm{H}^3(\Delta/p)^{\varphi = d^2}) < \mathrm{rank}_{\mathcal{O}_{\mathbb{C}_p^\flat}}(\mathrm{H}^3(\Delta/p))$$

it seems that some condition on p are necessary. This seems compatible with the results in [FKW24].

5.3. **Proof of Theorem 1.2.1.** This follows from Theorems 1.3.1, 1.3.2 and 1.3.3.

6. Abelian varieties and Kummer varieties

In this section we specialise to the case in which X is an abelian variety. We start in Section 6.1 by recalling the relationship between prismatic cohomology of abelian varieties, their p-torsion subgroups and Dieudonné theory. We then apply this to the products of elliptic curves (Section 6.2), to abelian varieties of positive p-rank and to their associated Kummer varieties (Section 6.3).

If G is a finite locally free group scheme over $\mathcal{O}_{\mathbb{C}_p}$ we write

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0$$

for the étale connected sequence of G, and we denote by $G^\vee$ the Cartier dual of G.

6.1. **Prismatic Dieudonné theory.** Let $\mathcal{C}$ be the category of free $\mathcal{O}_{\mathbb{C}_p^\flat}$-modules M endowed with a φ-linear Frobenius $\varphi_M : M \to M$ which becomes an isomorphism after inverting d. The following theorem summarises the main results we need from [AL23].

Theorem 6.1.1. *There exists an exact contravariant fully faithful functor*

$$\mathbb{D} : \{p\text{-torsion finite locally free } \mathcal{O}_{\mathbb{C}_p^\flat}\text{- group schemes}\} \to \mathcal{C}$$

such that

(1) $\mathbb{D}(\mu_p) = \mathcal{O}(-1)$;
(2) $\mathbb{D}(\mathcal{G}^\vee) \simeq \mathbb{D}(\mathcal{G})^\vee(-1)$;
(3) $\mathbb{D}(\mathbb{Z}/p) = \mathcal{O}$;
(4) *If* $\mathcal{A}/\mathcal{O}_{\mathbb{C}_p}$ *is an abelian scheme of dimension* g, *then* $\mathrm{H}^n(\mathcal{A}, \Delta/p) \simeq \wedge^n \mathbb{D}(\mathcal{A}[p])$ *and it has rank* $\binom{2g}{n}$;
(5) $\mathbb{D}(G)^{\varphi=1} \otimes \mathcal{O}_{\mathbb{C}_p^\flat} = \mathbb{D}(G^{\acute{e}t})$.

Proof. The existence of the functor follows from [AL23, Theorem 5.1.4] and the fact that a p-torsion A_{inf}-module has projective dimension ≤ 1 if and only if it is locally free as $\mathcal{O}_{\mathbb{C}_p^\flat}$-module. Then (1) follows from [AL23, Proposition 4.7.3 and the discussion before], since the map $\mathbb{Z}_p[[q-1]] \to A_{\mathrm{inf}}$ sending q to $[\epsilon^{1/p}]$ satisfies $f(\frac{q^p-1}{q-1}) = \xi$, see also [Mon22, Corollary 1.3] and [BL22, Notation 2.6.3]. Point (2) is the combination of point (1) and [AL23, Proposition 4.6.9], while (3) follows from (1) and (2). Finally (4) is [AL23, Corollaries 4.5.7 and 4.5.8] and (5) follows from [AL23, Remark 4.9.6]. □

6.2. **Products of elliptic curves.**

Proof of Proposition 1.5.1. Recall that the Künneth formula for étale cohomology

$$\mathrm{H}^2_{\acute{e}t}(X_{\mathbb{C}_p}, \mathbb{Z}/p) \simeq \mathrm{H}^2_{\acute{e}t}(Z_{\mathbb{C}_p}, \mathbb{Z}/p) \oplus \mathrm{H}^2_{\acute{e}t}(W_{\mathbb{C}_p}, \mathbb{Z}/p) \oplus \mathrm{H}^1_{\acute{e}t}(Z_{\mathbb{C}_p}, \mathbb{Z}/p) \otimes \mathrm{H}^1_{\acute{e}t}(W_{\mathbb{C}_p}, \mathbb{Z}/p) \simeq$$

$$\mathbb{Z}/p \oplus \mathbb{Z}/p \oplus \mathrm{Hom}_{\mathbb{C}_p}(Z[p], W[p])$$

induces, since $\mathrm{NS}(X_{\mathbb{C}_p}) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Hom}_{\mathbb{C}_p}(Z, W)$, a natural isomorphism

$$\mathrm{Br}(X_{\mathbb{C}_p})[p] \simeq \mathrm{Hom}_{\mathbb{C}_p}(Z[p], W[p])/\mathrm{Hom}_{\mathbb{C}_p}(Z, W).$$

Since $\mathrm{H}^i(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$ and $\mathrm{H}^i(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$ are torsion-free by Proposition 3.2.2, the Künneth formula for prismatic cohomology (see e.g. [AL23, Corollary 3.5.2]) gives an isomorphism

$$\begin{aligned}\mathrm{H}^2(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) &\simeq \mathrm{H}^2(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \oplus \mathrm{H}^2(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \oplus \mathrm{H}^1(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \otimes \mathrm{H}^1(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)\\ &\simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathrm{H}^1(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \otimes \mathrm{H}^1(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p).\end{aligned}$$

Hence, by Theorems 1.3.1 and 1.3.2, it is enough to show that

$$(\mathrm{H}^1(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \otimes \mathrm{H}^1(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p))^{\varphi=d} \simeq \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p], \mathcal{W}[p]).$$

By Theorem 6.1.1, this is equivalent to show that

$$(\mathrm{H}^1(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \otimes \mathrm{H}^1(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p))^{\varphi=d} \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{D}(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p]), \mathbb{D}(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p])).$$

For this, observe that there are natural isomorphisms compatible with the Frobenius

$$\begin{aligned} \mathrm{H}^1(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) \otimes \mathrm{H}^1(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p) &\simeq \mathrm{Hom}(\mathrm{H}^1(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)^\vee, \mathrm{H}^1(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)) \\ &\simeq \mathrm{Hom}(\mathbb{D}(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p])^\vee, \mathbb{D}(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p])) \\ &\simeq \mathrm{Hom}(\mathbb{D}(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p]^\vee)(1), \mathbb{D}(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p])) \\ &\simeq \mathrm{Hom}(\mathbb{D}(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p])(1), \mathbb{D}(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p])), \end{aligned}$$

where the first isomorphism follows from the fact that $\mathrm{H}^1(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$ and $\mathrm{H}^1(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)$ are finite free, the second and the third from Theorem 6.1.1 and the last from the fact that $\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p]$ is self dual. To conclude, observe that

$$\begin{aligned} (\mathrm{Hom}(\mathbb{D}(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p])(1), \mathbb{D}(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p])))^{\varphi=d} &= (\mathrm{Hom}(\mathbb{D}(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p]), \mathbb{D}(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p])))^{\varphi=1} \\ &= \mathrm{Hom}_{\mathcal{C}}(\mathbb{D}(\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p]), \mathbb{D}(\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p])). \qquad \square \end{aligned}$$

The next corollary concerning fields of definition of interesting Brauer classes follows immediately from Proposition 1.5.1.

Corollary 6.2.1. *Let $X = Z \times W$ for elliptic curves Z, W and let L/K be a field extension with $\mathrm{Hom}_L(Z[p], W[p]) = \mathrm{Hom}_{\mathbb{C}_p}(Z[p], W[p])$. Then the natural map*

$$\operatorname{Br} X_L[p]/\operatorname{fil}_0(X_L) \to \operatorname{Br}(X_{\overline{K}})[p]/\operatorname{Br}(X_{\overline{K}})^{gb}[p]$$

is surjective.

Remark 6.2.2. In Corollary 6.2.1, one can take $L = K(Z[p], W[p])$, for example. For elliptic curves with complex multiplication by the ring of integers of an imaginary quadratic field F, this is related to the ray class field of F with modulus p. Furthermore, if the CM field satisfies $\mathcal{O}_F^\times = \{\pm 1\}$, then [New16, Proposition 2.2] shows that it suffices to take L to be the compositum of K with the ring class field of conductor p, an extension of F of degree $h_F \cdot \left(p - \left(\frac{\delta_F}{p}\right)\right)$, where h_F is the class number, δ_F is the discriminant, and $\left(\frac{\delta_F}{p}\right)$ is the Legendre symbol for p odd and the Kronecker symbol for $p = 2$.

Corollary 6.2.3. *Let $X = Z \times W$ for elliptic curves Z, W with good reduction and write*

$$\mathrm{Ker}_{\mathcal{Z}} := \mathrm{Ker}\Big(Z[p](\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{Z}[p](\bar{k})\Big), \quad \mathrm{Ker}_{\mathcal{W}} := \mathrm{Ker}\Big(W[p](\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{W}[p](\bar{k})\Big).$$

(1) Assume that the special fibres of Z, W are ordinary. Then $\operatorname{Br}(X_{\overline{K}})[p]/\operatorname{Br}(X_{\overline{K}})^{gb}[p]$ is one-dimensional, generated by any homomorphism sending an element in $\mathrm{Ker}_{\mathcal{Z}}$ to an element not in $\mathrm{Ker}_{\mathcal{W}}$.
(2) Assume that the special fibre of W is ordinary but that of Z is supersingular. Then $\operatorname{Br}(X_{\overline{K}})[p]/\operatorname{Br}(X_{\overline{K}})^{gb}[p]$ is two-dimensional, generated by the homomorphisms $Z[p] \to W[p]$ whose image is not contained in $\mathrm{Ker}_{\mathcal{W}}$.
(3) Assume that $Z = W$ and that the special fibre is supersingular. Then any element in $\mathrm{End}_{\mathbb{C}_p}(Z[p])$ whose characteristic polynomial has two distinct roots in $\mathbb{F}_p$ gives a non-zero element in $\operatorname{Br}(X_{\overline{K}})[p]/\operatorname{Br}(X_{\overline{K}})^{gb}[p]$. In particular, the dimension of $\operatorname{Br}(X_{\overline{K}})[p]/\operatorname{Br}(X_{\overline{K}})^{gb}[p]$ is at least one.

Proof.
(1) Since the special fibres of Z, W are ordinary,

$$\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p] \simeq \mu_p \times \mathbb{Z}/p \simeq \mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p].$$

Hence $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p], \mathcal{W}[p])$ can be written as,

$$\mathrm{End}(\mathbb{Z}/p) \oplus \mathrm{End}(\mu_p) \oplus \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathbb{Z}/p, \mu_p) \oplus \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mu_p, \mathbb{Z}/p).$$

Observe that $\mathrm{End}_{\mathcal{O}_{\mathbb{C}_p}}(\mathbb{Z}/p) \simeq \mathrm{End}_{\mathcal{O}_{\mathbb{C}_p}}(\mu_p) \simeq \mathbb{Z}/p$, generated by the identity. Moreover, also $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathbb{Z}/p, \mu_p) \simeq \mathbb{Z}/p$, generated by the morphism sending 1 to a primitive p^{th}-root of unity. By contrast, $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mu_p, \mathbb{Z}/p) = 0$, since μ_p is connected and $\mathbb{Z}/p$ is totally disconnected. Since $\mathrm{Ker}\Big(\mathcal{Z}[p](\mathcal{O}_{\mathbb{C}_p})) \to \mathcal{Z}[p](\bar{k})\Big) = \mu_p(\mathbb{C}_p)$ and similarly for $\mathcal{W}$, Proposition 1.5.1 shows that $\mathrm{Br}(X_{\overline{K}})[p]/\mathrm{Br}(X_{\overline{K}})^{gb}[p]$ is one-dimensional, generated by any homomorphism sending an element in $\mathrm{Ker}\Big(Z[p](\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{Z}[p](\bar{k})\Big)$ to an element not in $\mathrm{Ker}_{\mathcal{W}}$.

(2) Since the special fibre of W is ordinary but that of Z is not,

$$\mathcal{W}_{\mathcal{O}_{\mathbb{C}_p}}[p] \simeq \mu_p \times \mathbb{Z}/p, \quad \text{while } \mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p] \text{ is connected.}$$

Hence,

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p], \mathcal{W}[p]) \simeq \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p], \mathbb{Z}/p) \oplus \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p], \mu_p).$$

Since $\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p]$ is connected, $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p], \mathbb{Z}/p) = 0$, while

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p], \mu_p) = \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathbb{Z}/p, \mathcal{Z}[p]^\vee) \simeq Z[p]^\vee(\mathcal{O}_{\mathbb{C}_p}) \simeq (\mathbb{Z}/p\mathbb{Z})^2.$$

Since $\mathrm{Ker}\Big(W[p](\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{W}[p](\bar{k})\Big) = \mu_p(\mathcal{O}_{\mathbb{C}_p})$, Proposition 1.5.1 shows that the quotient $\mathrm{Br}(X_{\overline{K}})[p]/\mathrm{Br}(X_{\overline{K}})^{gb}[p]$ is two-dimensional, generated by the homomorphisms $Z[p] \to W[p]$ whose image is not contained in $\mathrm{Ker}_{\mathcal{W}}$.

(3) Since Z has supersingular reduction, $\mathcal{Z}[p]_{\bar{k}}$ is not the product of two subgroups, hence also $\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p]$ is not. In particular, $\mathrm{End}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p])$ contains no idempotents apart from 0 and 1. Hence, all non-zero multiples of the idempotent elements different from 0 and 1 in $\mathrm{End}_{\mathbb{C}_p}(Z[p])$ give non-zero elements in the quotient $\mathrm{Br}(X_{\overline{K}})[p]/\mathrm{Br}(X_{\overline{K}})^{gb}[p]$. Since the scalars in $\mathrm{End}_{\mathbb{C}_p}(Z[p]) \simeq M_2(\mathbb{F}_p)$ correspond to elements in $\mathrm{Br}(X_{\overline{K}})^{gb}[p]$, any element in $\mathrm{End}_{\mathbb{C}_p}(Z[p])$ whose characteristic polynomial has two distinct roots in $\mathbb{F}_p$ gives a non-zero element in $\mathrm{Br}(X_{\overline{K}})[p]/\mathrm{Br}(X_{\overline{K}})^{gb}[p]$. □

Remark 6.2.4. One can also obtain a result similar to Corollary 6.2.3(3) given an isogeny of degree coprime to p between elliptic curves Z and W with supersingular reduction.

Remark 6.2.5. Corollary 6.2.3 sheds light on existing results in the literature. For example, Corollary 6.2.3(1) explains the existence of the 3-torsion arithmetically interesting Brauer class in [Pag25, Example 4.12] and shows that all 3-torsion classes with non-constant evaluation maps at primes above 3 are scalar multiples of this one. See also Section 6.2.2.

6.2.1. *The CM case.* We now specialise to the situation in which $X = Z \times Z$ for an elliptic curve Z with complex multiplication (CM) by the ring of integers $\mathcal{O}_L$ of an imaginary quadratic number field L. Writing δ for the discriminant of $\mathbb{Q}(\sqrt{-d})$, a generator for the ring of integers of $\mathbb{Q}(\sqrt{-d})$ is given by $\gamma = (\delta + \sqrt{\delta})/2$. Fix a $\mathbb{Z}/p$-basis for $Z[p](\mathbb{C}_p)$ of the form $P, \gamma P$ for some $P \in Z[p](\mathbb{C}_p)$. With respect to this basis, multiplication by γ on $Z[p](\mathbb{C}_p)$ is given by the following matrix with entries in $\mathbb{Z}/p$:

$$\begin{pmatrix} 0 & \frac{\delta(1-\delta)}{4} \\ 1 & \delta \end{pmatrix}.$$

(Note that $\frac{\delta(1-\delta)}{4} \in \mathbb{Z}$ so it has a well-defined image in $\mathbb{Z}/p$ for all primes p, including $p = 2$.)

Corollary 6.2.6. *Let $X = Z \times Z$ for an elliptic curve Z with CM by $\mathcal{O}_L = \mathbb{Z}[\gamma]$.*

(1) Suppose that $p \neq 2$ and let $\sigma \in \mathrm{End}_{\mathbb{C}_p}(Z[p])$ be induced by complex conjugation. Then σ corresponds to a non-zero element in $\mathrm{Br}(X_{\overline{K}})[p]/\mathrm{Br}(X_{\overline{K}})^{gb}[p]$.

(2) Suppose that Z has supersingular reduction. If p is odd or $p \nmid \delta$ then $\mathrm{Br}(X_{\overline{K}})^{gb}[p]$ is trivial. Consequently, the dimension of $\mathrm{Br}(X_{\overline{K}})[p]/\mathrm{Br}(X_{\overline{K}})^{gb}[p]$ is two.

Proof.

(1) If Z has supersingular reduction, then this follows from Corollary 6.2.3, since $\sigma^2 = 1$ and $p \neq 2$ (so $\sigma \neq 1$). Now suppose that Z has ordinary reduction. By Corollary 6.2.3, since

$$\mathrm{Ker}\Big(\mathcal{Z}[p](\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{Z}[p](\bar{k})\Big) = \mu_p(\mathbb{C}_p)$$

is one-dimensional and stable by all the endomorphisms of $\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p]$, it is enough to show that $\mu_p(\mathbb{C}_p)$ is not stable under the action of σ. The group $\mu_p(\mathbb{C}_p)$ is stable by the action of γ, hence it is an eigenspace for γ. Since Z has ordinary reduction, δ is a square modulo p. Since the eigenvalues of γ are $\delta \pm \sqrt{\delta}$ and δ is negative, the eigenvectors for γ are swapped by the action of σ. Consequently, $\mu_p(\mathbb{C}_p)$ is not preserved by the action of σ and therefore σ does not lift to an endomorphism of $\mathcal{Z}_{\mathcal{O}_{\mathbb{C}_p}}[p]$.

(2) Let e_1, e_2 be idempotent endomorphisms of $Z_{\mathbb{C}_p}[p]$ defined by the following matrices with respect to our fixed basis:

$$e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since $e_1, e_2, \gamma, \mathrm{Id}$ is a basis of $\mathrm{End}_{\mathbb{C}_p}(Z[p])$ and γ, Id is a basis of $\mathrm{End}_{\mathbb{C}_p}(Z)$, it suffices to prove that all non-zero linear combinations $ae_1 + be_2 \in \mathrm{End}_{\mathbb{C}_p}(Z[p])$, with $a, b \in \mathbb{F}_p$, do not lift to $\mathrm{End}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p])$. The characteristic polynomial of $ae_1 + be_2$ is $X(X - a - b)$ so if $a + b \neq 0$, the result follows from Corollary 6.2.3. Now suppose that $a + b = 0$, so $\theta = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Note that θ lifts to $\mathrm{End}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p])$ if and only if $\theta + t\gamma$ lifts to $\mathrm{End}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{Z}[p])$ for all $t \in \mathbb{F}_p$. Now, the characteristic polynomial of $\theta + t$ is

$$f(X) = X(X - t\delta) - t\left(b + t\left(\frac{\delta(1-\delta)}{4}\right)\right).$$

If we can find $t \in \mathbb{F}_p$ such that $f(X)$ splits into two linear factors over $\mathbb{F}_p$, then we are done by Corollary 6.2.3. A theorem of Shimura and Taniyama states that E has supersingular reduction if and only if p does not split in $\mathbb{Q}(\sqrt{-d})/\mathbb{Q}$. If p ramifies in $\mathbb{Q}(\sqrt{-d})/\mathbb{Q}$ then $p \mid \delta$ (whereby p is odd by assumption) and $f(X)$ becomes $X^2 - tb$. If we take $t = b$ then $f(X)$ splits, as required. So we may assume that p does not ramify and must therefore be inert in $\mathbb{Q}(\sqrt{-d})/\mathbb{Q}$. Therefore, p does not divide $\frac{1-\delta}{4}$ and we can take $t \in \mathbb{F}_p^\times$ such that $b + t\left(\frac{\delta(1-\delta)}{4}\right) = 0$. Then $f(X)$ becomes $X(X - t\delta)$, which is split.

The last assertion follows immediately as $\mathrm{Br}(X_{\overline{K}})[p] \cong (\mathbb{Z}/p)^2$ by work of Grothendieck, see [CS21, Proposition 5.2.9]. □

Remark 6.2.7. If $X = Z_1 \times Z_2$ where Z_1, Z_2 are elliptic curves with CM by orders $\mathcal{O}_1, \mathcal{O}_2$ in L, then Corollary 6.2.6 applies with the additional constraint that the prime p be coprime to $[\mathcal{O}_L : \mathcal{O}_i]$ for $i = 1, 2$.

6.2.2. *Comparison with previous work.* Products of CM elliptic curves (and their associated Kummer surfaces) have been the source of many examples of transcendental Brauer classes obstructing weak approximation in the literature to date. The results in [IS15; IS17] are obtained by relating diagonal quartic surfaces to products of curves with CM by $\mathbb{Z}[i]$. Products of elliptic curves over $\mathbb{Q}$ with CM by other maximal orders have been studied in [New16; AN25]. All these examples of transcendental Brauer classes obstructing weak approximation are related to complex conjugation as in Corollary 6.2.6(1), possibly after composing with a geometric isomorphism $\phi : Z_{\bar{\mathbb{Q}}} \to W_{\bar{\mathbb{Q}}}$. This is spelt out in [AN25, Proposition 4.6] and [Ala24, Lemma 2.3.3], for example.

Note that in the examples in [IS15; IS17; New16; AN25], the relevant Brauer classes have order p where p is an odd prime of bad, but potentially good, reduction for the abelian surface. In the good reduction setting, Corollary 6.2.6(1) shows that p-torsion Brauer classes coming from complex conjugation have non-constant (in fact, surjective – see Proposition 2.2.3) evaluation maps at primes above p over all finite extensions. In the ordinary reduction cases, (including $p = 5$ in [IS15, Theorem 1.1], all cases of [AN25, Theorem 1.3], and the case $\ell = 7$ of [AN25, Theorem 1.4]), Corollary 6.2.3(1) explains why the only arithmetically interesting Brauer classes come from (scalar multiples of) complex conjugation. Moreover, Corollary 6.2.6 shows that the constant evaluation map of the element $\mathcal{A} \in \mathrm{Br}(D)[3]$ for D of type I in [IS17, Theorem 2.3] is merely a temporary phenomenon – the evaluation map will become surjective after passing to a finite extension where the surface attains good reduction above 3. This example shows that good reduction is a necessary hypothesis in Proposition 2.2.3.

6.3. **Abelian varieties of positive p-rank and associated Kummer varieties.** In this section we prove Theorem 1.5.3. Let $K/\mathbb{Q}_p$ be a p-adic field and X an abelian variety of dimension $g \geq 2$ with good reduction. Following [SZ17], given a k-torsor T for the k-group scheme $X[2]$, we define the associated 2-covering of X as the quotient $Y := (X \times_k T)/X[2]$ by the diagonal action of $X[2]$. The antipodal involution ι_X on X induces an involution $\iota_Y \colon Y \to Y$. Let $\sigma \colon Y' \to Y$ be the blowing-up of the 2^{2g}-point closed subscheme $T \subseteq Y$. The involution ι_Y preserves T and so gives rise to an involution $\iota_{Y'}$ on Y'. The Kummer variety attached to Y is defined as the quotient $Y'/\iota_{Y'} =: \mathrm{Kum}(X_T)$ and it is a smooth proper variety. If p is odd then $\mathrm{Kum}(X_T)$ has good reduction, since X has good reduction.

We write $\pi \colon Y' \to \mathrm{Kum}(X_T)$ for the double covering whose branch locus is $E := \sigma^{-1}(T)$. In [SZ17, Proposition 2.7], Skorobogatov and Zarhin prove that the morphisms π and σ induce an isomorphism of Γ-modules

$$\varphi \colon \mathrm{Br}(\mathrm{Kum}(X_T)_{\bar{K}})[p] \to \mathrm{Br}(X_{\bar{K}})[p].$$

It may be useful to note that φ induces an injection $\mathrm{Br}(\mathrm{Kum}(X_T)_{\bar{K}})^{gb}[p] \hookrightarrow \mathrm{Br}(X_{\overline{K}})^{gb}[p]$, by Theorem 1.3.1.

Proof of Theorem 1.5.3. We start with the part of the statement on abelian varieties. By Theorems 1.3.1 and 1.3.2, it is enough to show that

$$\mathrm{rank}_{\mathcal{O}_{\mathbb{C}_p^\flat}}(\mathrm{H}^2(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)) - \dim_{\mathbb{F}_p}(\mathrm{H}^2(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_p}}, \Delta/p)^{\varphi-d}) \geq 2g - 1 - e.$$

Since $e > 0$, there is a surjection $\mathcal{X}[p] \to \mathbb{Z}/p$ with finite locally free kernel. Since $\mathcal{X}[p]$ is self-dual, there is a short exact sequence of finite locally free group schemes

$$0 \to \mu_p \to \mathcal{X}[p] \to J \to 0,$$

such that $\mathrm{rank}(J^{\text{ét}}) = e$. By Theorem 6.1.1, this induces a short exact sequence

$$0 \to \mathbb{D}(J) \to \mathbb{D}(\mathcal{X}[p]) \to \mathcal{O}(-1) \to 0$$

in $\mathcal{C}$. By applying $\wedge^2$ and using Theorem 6.1.1, we get an exact sequence

$$0 \to \wedge^2\mathbb{D}(J) \to \mathrm{H}^2(\Delta/p) \to \mathbb{D}(J) \otimes \mathcal{O}(-1) \to 0.$$

Hence it is enough to show that

$$\dim_{\mathbb{F}_p}((\mathbb{D}(J) \otimes \mathcal{O}(-1))^{\varphi-d}) = e$$

But

$$\dim_{\mathbb{F}_p}((\mathbb{D}(J) \otimes \mathcal{O}(-1))^{\varphi-d}) = \dim_{\mathbb{F}_p}(\mathbb{D}(J)^{\varphi-1}) \overset{(*)}{=} \mathrm{rank}(J^{\text{ét}}) = e$$

where (*) follows from Theorem 6.1.1.

We now move to the Kummer variety associated to the $X[2]$-torsor T. By Theorem 1.3.1 it is enough to prove that

$$\mathrm{Im}(\mathrm{H}^2(\mathrm{Kum}(X_T)_{\bar{K}}, \mathbb{Z}/p) \to \mathrm{H}^2(\bar{K}(\mathrm{Kum}(X_T))^{\mathrm{sh}}, \mathbb{Z}/p)) \simeq \mathrm{Im}(\mathrm{H}^2(X_{\bar{K}}, \mathbb{Z}/p) \to \mathrm{H}^2(\bar{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p)).$$

As already pointed out at the beginning of Section 2.2 these maps factor through the p-torsion of the Brauer group. The result follows from the fact that $\bar{K}(\mathrm{Kum}(X_T))/\bar{K}(X)$ is a degree 2 extension and hence the map $\mathrm{H}^2(\bar{K}(\mathrm{Kum}(X_T))^{\mathrm{sh}}, \mathbb{Z}/p)) \to \mathrm{H}^2(\bar{K}(X)^{\mathrm{sh}}, \mathbb{Z}/p))$ is injective for $p > 2$. □

Remark 6.3.1. The problem of constructing even a single transcendental Brauer class giving an obstruction has hitherto been considered something of a challenge. Our results in the non-ordinary reduction setting give various instances of K3 surfaces having at least two independent transcendental elements of odd order p that play a role in the Brauer–Manin obstruction to weak approximation. For example, one can take X to be an abelian surface with p-rank one in Theorem 1.5.3, or alternatively consider $\mathrm{Kum}(Z \times Z)$ where Z is a CM elliptic curve with supersingular reduction and apply Corollary 6.2.6(2) together with the last part of the proof of Theorem 1.5.3.

References

[Ala24] Mohamed Alaa Tawfik. "Brauer–Manin obstructions for Kummer surfaces of products of CM elliptic curves". Available at https://kclpure.kcl.ac.uk/ws/portalfiles/portal/318878273/2024_Tawfik_Mohamed_21124970_ethesis.pdf. PhD thesis. King's College London, Nov. 2024.

[AN25] Mohamed Alaa Tawfik and Rachel Newton. "Transcendental Brauer-Manin obstructions on singular K3 surfaces". In: *Res. Number Theory* 11.1 (2025), Paper No. 16, 32.

[AL23] Johannes Anschütz and Arthur-César Le Bras. "Prismatic Dieudonné theory". In: *Forum Math. Pi* 11 (2023), Paper No. e2, 92.

[BV20] Jennifer Berg and Anthony Várilly-Alvarado. "Odd order obstructions to the Hasse principle on general K3 surfaces". In: *Math. Comp.* 89.323 (2020), pp. 1395–1416.

[Bha17] Bhargav Bhatt. *Lecture notes for a class on perfectoid spaces*. 2017.

[BL22] Bhargav Bhatt and Jacob Lurie. *Absolute prismatic cohomology*. 2022. arXiv: 2201.06120.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310.

[BS22] Bhargav Bhatt and Peter Scholze. "Prisms and prismatic cohomology". In: *Ann. of Math. (2)* 196.3 (2022), pp. 1135–1275.

[BK86] Spencer Bloch and Kazuya Kato. "p-adic étale cohomology". In: *Inst. Hautes Études Sci. Publ. Math.* 63 (1986), pp. 107–152.

[BN23] Martin Bright and Rachel Newton. "Evaluating the wild Brauer group". In: *Invent. Math.* 234.2 (2023), pp. 819–891.

[CS21] Jean-Louis Colliot-Thélène and Alexei N. Skorobogatov. *The Brauer-Grothendieck group*. Vol. 71. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham, [2021] ©2021, pp. xv+453.

[Elk87] Noam D. Elkies. "The existence of infinitely many supersingular primes for every elliptic curve over **Q**". In: *Invent. Math.* 89.3 (1987), pp. 561–567.

[FKW24] Benson Farb, Mark Kisin, and Jesse Wolfson. "Essential dimension via prismatic cohomology". In: *Duke Math. J.* 173.15 (2024), pp. 3059–3106.

[HVV11] Brendan Hassett, Anthony Várilly-Alvarado, and Patrick Varilly. "Transcendental obstructions to weak approximation on general K3 surfaces". In: *Adv. Math.* 228.3 (2011), pp. 1377–1404.

[Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996, pp. x+450.

[Ier10] Evis Ieronymou. "Diagonal quartic surfaces and transcendental elements of the Brauer groups". In: *J. Inst. Math. Jussieu* 9.4 (2010), pp. 769–798.

[IS15] Evis Ieronymou and Alexei N. Skorobogatov. "Odd order Brauer-Manin obstruction on diagonal quartic surfaces". In: *Adv. Math.* 270 (2015), pp. 181–205.

[IS17] Evis Ieronymou and Alexei N. Skorobogatov. "Corrigendum to "Odd order Brauer-Manin obstruction on diagonal quartic surfaces" [Adv. Math. 270 (2015) 181–205] [MR3286534]". In: *Adv. Math.* 307 (2017), pp. 1372–1377.

[Kat89] Kazuya Kato. "Swan conductors for characters of degree one in the imperfect residue field case". In: *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*. Vol. 83. Contemp. Math. Amer. Math. Soc., Providence, RI, 1989, pp. 101–131.

[Kat79] Nicholas M. Katz. "Slope filtration of F-crystals". In: *Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I*. Vol. 63. Astérisque. Soc. Math. France, Paris, 1979, pp. 113–163.

[LL23] Shizhang Li and Tong Liu. "On the u^∞-torsion submodule of prismatic cohomology". In: *Compos. Math.* 159.8 (2023), pp. 1607–1672.

[Mil16] James S. Milne. *Étale Cohomology (PMS-33)*. Princeton Mathematical Series v. 33. Princeton University Press, 2016.

[Mon22] Shubhodip Mondal. "A computation of prismatic Dieudonné module". In: *Acta Arith.* 205.1 (2022), pp. 87–96.

[Mor18] Matthew Morrow. "p-adic vanishing cycles as Frobenius-fixed points". In: *Preprint* (2018).

[New16] Rachel Newton. "Transcendental Brauer groups of products of CM elliptic curves". In: *J. Lond. Math. Soc. (2)* 93.2 (2016), pp. 397–419.

[New24] Rachel Newton. "Corrigendum: Transcendental Brauer groups of products of CM elliptic curves". In: *J. Lond. Math. Soc. (2)* 93.2 (2024), pp. 397–419.

[Pag25] Margherita Pagano. "The role of primes of good reduction in the Brauer-Manin obstruction". In: *To appear in Algebra and Number Theory* (2025). arXiv: 2307.16030 [math.NT].

[Pre13] Thomas Preu. "Example of a transcendental 3-torsion Brauer-Manin obstruction on a diagonal quartic surface". In: *Torsors, étale homotopy and applications to rational points*. Vol. 405. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2013, pp. 447–459.

[SZ17] Alexei N. Skorobogatov and Yuri G. Zarhin. "Kummer varieties and their Brauer groups". In: *Pure Appl. Math. Q.* 13.2 (2017), pp. 337–368.

[Sta25] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2025.

[TZ23] Koshikawa Teruhisa and Yao Zijian. *Logarithmic prismatic cohomology II*. 2023. arXiv: 2306.00364.

[Vak25] Ravi Vakil. *The Rising Sea, foundation of algebraic geometry*. 2025.

[Wit04] Olivier Wittenberg. "Transcendental Brauer-Manin obstruction on a pencil of elliptic curves". In: *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*. Vol. 226. Progr. Math. Birkhäuser Boston, Boston, MA, 2004, pp. 259–267.

[Yao23] Zijian Yao. "Frobenius and the Hodge filtration of the generic fiber". In: *Int. Math. Res. Not. IMRN* 12 (2023), pp. 10156–10173.

Institut de Recherche Mathématique Avancée (IRMA), Université de Strasbourg, Strasbourg, France
Email address: eambrosi@unistra.fr

Department of Mathematics, King's College London, Strand, London, UK
Email address: rachel.newton@kcl.ac.uk

Department of Mathematics, South Kensington Campus, Imperial College London, London, UK
Email address: m.pagano@imperial.ac.uk