Topological groupoids with involution and real algebraic stacks

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Abstract

To a topological groupoid endowed with an involution, we associate a topological groupoid of fixed points, generalizing the fixed-point subspace of a topological space with involution. We prove that when the topological groupoid with involution arises from a Deligne-Mumford stack over $\mathbb{R}$, this fixed locus coincides with the real locus of the stack. This provides a topological framework to study real algebraic stacks, and in particular real moduli spaces. Finally, we propose a Smith–Thom type conjecture in this setting, generalizing the Smith–Thom inequality for topological spaces endowed with an involution.

1 Introduction

This work is the first in a two-part series devoted to the study of the topology of real algebraic stacks. In this paper, which takes a more topological perspective, we study the topology of real algebraic stacks via their associated topological groupoids with involution, and establish several general results in this setting. We also formulate a Smith–Thom type conjecture for such groupoids.

The second paper of this series, see [AGF25], adopts a more algebraic point of view. There, we develop techniques for computing the topology of various real Deligne–Mumford stacks – such as finite quotient stacks and gerbes over a real variety – and use these techniques to verify the conjecture in several cases.

1.1 The topology of real moduli spaces. Our interest in topological groupoids with involution is motivated by the study of the topology of moduli spaces over $\mathbb{R}$, see e.g. [GH81; SS89; Eti+10] and [GF22b; GF24]. Recently, there has been growing interest in understanding the cohomology of the real locus $M(\mathbb{R})$ of the associated coarse moduli space M, see e.g. [Fra18; BS22; Fu23; KR24]. However, such a study says something about the *real moduli space* associated to the moduli problem only when this real moduli space coincides with the real locus of the coarse moduli space – a phenomenon that fails in general (think of moduli of abelian varieties or curves).

To overcome this limitation, one is naturally led to move beyond the setting of real algebraic varieties (and hence of topological spaces with involution) and into the realm of real algebraic stacks (and hence of topological groupoids with involution). Similar to the way in which purely topological methods often play a crucial role in the study of the real locus $X(\mathbb{R})$ of a real algebraic variety X, the study of real algebraic stacks is facilitated by working within the context of topological groupoids with involution.

1.2 Topological groupoids with involution. Recall that a *topological groupoid* $\mathscr{X} = [R \rightrightarrows U]$ consists of two topological spaces, U (the space of objects) and R (the space of arrows), and a collection of continuous maps (s, t, c, e, i) satisfying a number of natural conditions (see Section 3.1 for more details). For instance, $s\colon R \to U$ is the source map, $t\colon R \to U$ the target map, and $c\colon R \times_{s,U,t} R \to R$ the composition map. If $f \in R$ is such that $s(f) = x$ and $t(f) = y$, we write $x \xrightarrow{f} y$ and we say that f is an isomorphism between x and y. If $f, g \in R$ are such that $t(f) = s(g)$ we write $g \circ f$ for $c(g, f)$.

Write $x \cong y$ if there exists a $f \in R$ with $x \xrightarrow{f} y$. The axioms of topological groupoids ensure that $\cong$ is an equivalence relation, so that we can consider $|\mathscr{X}| = U/_{\cong}$, the set of isomorphism classes of objects of $\mathscr{X}$. We equip $|\mathscr{X}|$ with the quotient topology.

Example 1.1. Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{C}$. Let $U \to \mathcal{X}$ be a surjective étale presentation by a scheme U, and let R be a scheme with $R \cong U \times_{\mathcal{X}} U$. The two projections $\pi_1, \pi_2\colon (U \times_{\mathcal{X}} U)(\mathbb{C}) \to U(\mathbb{C})$ define maps $s, t\colon R(\mathbb{C}) \to U(\mathbb{C})$ that extend to the structure of a topological groupoid $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$. There is a natural bijection between $|\mathscr{X}|$ and the set $|\mathcal{X}(\mathbb{C})|$ of isomorphism classes of objects in $\mathcal{X}(\mathbb{C})$; see Construction 4.1 for more details.

An *involution* $\sigma\colon \mathscr{X} \to \mathscr{X}$ consists of involutions $\sigma\colon R \to R$ and $\sigma\colon U \to U$ that are compatible all the structure maps of the topological groupoid.

Example 1.2. Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$, let $U \to \mathcal{X}$ be a surjective étale presentation by a scheme U over $\mathbb{R}$. If $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ is the topological groupoid associated to $\mathcal{X}_{\mathbb{C}}$ as in Example 1.1, the natural anti-holomorphic involutions on $R(\mathbb{C})$ and $U(\mathbb{C})$ define an involution $\sigma\colon \mathscr{X} \to \mathscr{X}$.

Let $\mathscr{X} = [R \rightrightarrows U]$ be a topological groupoid, equipped with an involution $\sigma\colon \mathscr{X} \to \mathscr{X}$. Inspired by the case of topological spaces with involution, we call the pair $(\mathscr{X}, \sigma)$ a *topological G-groupoid*, where

$$G \coloneqq \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2.$$

We define the *fixed locus* of the topological G-groupoid $(\mathscr{X}, \sigma)$ as follows:

$$\left|\mathscr{X}^G\right| \coloneqq \left\{(x,\varphi) \in U \times R \text{ such that } x \xrightarrow{\varphi} \sigma(x) \text{ with } \sigma(\varphi)\circ\varphi = \mathrm{id}\right\}/_{\cong} \tag{1}$$

where $(x,\varphi) \cong (y,\psi)$ if there exists an arrow $f\colon x \to y$ in R such that $\psi \circ f = \sigma(f)\circ\varphi$. This definition is motivated by the following example.

Example 1.3. Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$, and let $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ be the topological groupoid with involution associated to $\mathcal{X}$ as in Example 1.2. By the theory of descent (see [Gro60]), one has a canonical bijection $\left|\mathscr{X}^G\right| \cong |\mathcal{X}(\mathbb{R})|$. For details, see Lemma 4.4.

1.3 Topology of real Deligne–Mumford stacks. If $\mathcal{X}$ is a real Deligne–Mumford stack, one can define a natural topology on the set $|\mathcal{X}(\mathbb{R})|$ of isomorphism classes of the groupoid $\mathcal{X}(\mathbb{R})$, by choosing an étale presentation $U \to \mathcal{X}$ by a real scheme U such that $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective (cf. [GF22b, Theorem 7.4]); the resulting topology is independent of the choice of presentation (cf. [GF22b, Proposition 7.6]) and called the *real analytic topology* of $|\mathcal{X}(\mathbb{R})|$ (cf. [GF22b, Definition 7.5]).

Our first main result compares the real analytic topology of $|\mathcal{X}(\mathbb{R})|$ with the topology of the associated topological groupoid. For a topological groupoid $\mathscr{X} = [R \rightrightarrows U]$ equipped with an involution $\sigma\colon \mathscr{X} \to \mathscr{X}$, endow the fixed locus $|\mathscr{X}^G|$ with the quotient topology (it is a quotient of a subspace of $U \times R$, see equation (1) above).

Theorem 1.4. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$. Let $U \to \mathcal{X}$ be an étale surjective morphism from a scheme U over $\mathbb{R}$. Consider the associated topological groupoid $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ with involution $\sigma\colon \mathscr{X} \to \mathscr{X}$. When $\left|\mathscr{X}^G\right|$ is endowed with the quotient topology and $|\mathcal{X}(\mathbb{R})|$ with the real analytic topology, the bijection $\left|\mathscr{X}^G\right| \xrightarrow{\sim} |\mathcal{X}(\mathbb{R})|$ of Example 1.3 is a homeomorphism.*

As an application of Theorem 1.4, and to illustrate the value of topological groupoid techniques in the study of the topology of real algebraic stacks, we analyze the map on real loci $f_{\mathbb{R}}\colon |\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ induced by the coarse moduli space morphism $f\colon \mathcal{X} \to M$, where $\mathcal{X}$ is a separated Deligne–Mumford stack of finite type over $\mathbb{R}$ (recall that the existence of f is guaranteed by [KM97]). Our main result in this direction is as follows.

Theorem 1.5. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$, with coarse moduli space $f\colon \mathcal{X} \to M$. Assume $\#\mathrm{Aut}(x)$ is constant for $x \in \mathcal{X}(\mathbb{C})$. Then the map $f_{\mathbb{R}}\colon |\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ is open, and a topological covering over its image.*

Topological groupoid techniques play a key role in the proof of Theorem 1.5, as they allow one to work "euclidean locally" on $M(\mathbb{R})$, in analogy with the familiar approach of working analytically locally in the study of real algebraic varieties. Theorem 1.5 will be applied in the subsequent paper [AGF25] to study the topology of various types of real gerbes over a real variety.

1.4 Smith–Thom inequality for topological groupoids with involution. Recall that if T is a locally compact Hausdorff topological space such that $\dim \mathrm{H}^*(T, \mathbb{Z}/2)$ is finite, endowed with a G-action given by an involution $\sigma\colon T \to T$, the *Smith–Thom inequality* states that

$$\dim \mathrm{H}^*(T^G, \mathbb{Z}/2) \leq \dim \mathrm{H}^*(T, \mathbb{Z}/2). \tag{2}$$

See [Flo52; Bor60; Tho65; DIK00; Man17] for various proofs. The inequality (2) is particularly significant when $T = X(\mathbb{C})$ for a real algebraic variety X, and $\sigma\colon X(\mathbb{C}) \to X(\mathbb{C})$ is the anti-holomorphic involution given by complex conjugation, so that $T^G = X(\mathbb{R})$. In this setting, the inequality (2) provides an upper bound on the cohomology of $X(\mathbb{R})$ in terms of the cohomology of $X(\mathbb{C})$, usually easier to compute. As such, it stands as one of the foundational results in real algebraic geometry.

In light of Theorem 1.4, a natural first step toward understanding the topology of real moduli spaces is to ask whether an equality like (2) might hold for topological groupoids with an involution. Unfortunately, the naive inequality $\dim \mathrm{H}^*(|\mathscr{X}^G|, \mathbb{Z}/2) \leq \dim \mathrm{H}^*(|\mathscr{X}|, \mathbb{Z}/2)$ fails in general for such groupoids. For example, consider the classifying groupoid $\mathscr{X} \coloneqq [\Gamma \rightrightarrows \mathrm{pt}]$ attached to a finite group Γ, equipped with the trivial involution $\sigma = \mathrm{id}\colon \mathscr{X} \to \mathscr{X}$. In this case, $|\mathscr{X}| \simeq \mathrm{pt}$ while $|\mathscr{X}^G| \simeq \mathrm{H}^1(G, \Gamma)$ (cf. Example 3.6), so the inequality fails when $\#\mathrm{H}^1(G, \Gamma) > 1$ (e.g., when $\Gamma = \mathbb{Z}/2$).

This example shows that in a sense, the topological space $|\mathscr{X}|$ is too small to fully encode information about $|\mathscr{X}^G|$, as it does not capture the automorphisms of objects in $\mathscr{X}$. To take these into account, we consider the inertia inertia groupoid $\mathcal{I}_{\mathscr{X}} = [S \rightrightarrows R|_\Delta]$ whose objects are given by the space $R|_\Delta$ of arrows $\varphi \in R$ such that φ is an automorphism of $x = s(\varphi)$, and whose morphisms $\varphi \to \varphi'$ are given by morphisms $f\colon s(\varphi) \to s(\varphi')$ in R such that $\varphi' \circ f = f \circ \varphi$.

We conjecture the following generalization of the Smith–Thom inequality (2) to the setting of topological groupoids with involution.

Conjecture 1.6. *Let $\mathscr{X} = [R \rightrightarrows U]$ be a topological groupoid with involution $\sigma\colon \mathscr{X} \to \mathscr{X}$. Assume that R and U are locally compact and Hausdorff, and that the spaces $\left|\mathscr{X}^G\right|$ and $|\mathcal{I}_{\mathscr{X}}|$ have finite dimensional $\mathbb{Z}/2$-cohomology. Then, we have:*

$$\dim \mathrm{H}^*(\left|\mathscr{X}^G\right|, \mathbb{Z}/2) \leq \dim \mathrm{H}^*(|\mathcal{I}_{\mathscr{X}}|, \mathbb{Z}/2). \tag{3}$$

When $X = \mathscr{X}$ is a topological space, the natural map $|\mathcal{I}_{\mathscr{X}}| \to |\mathscr{X}|$ is a homeomorphism, hence (3) reduces to the usual Smith–Thom inequality (2). Moreover, the bound (if valid) is sharp; in fact, one can easily construct examples of topological groupoids with involution which are not topological spaces for which (3) is an equality.

Restricting Conjecture 1.6 to those topological groupoids with involution that arise from a real Deligne–Mumford stack, one obtains the following algebraic version.

Conjecture 1.7. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$. Let $I_{\mathcal{X}}$ be the coarse moduli space of the inertia stack of $\mathcal{X}$. Then the dimension of $\mathrm{H}^*(|\mathcal{X}(\mathbb{R})|, \mathbb{Z}/2)$ is less than or equal to the dimension of $\dim \mathrm{H}^*(I_{\mathcal{X}}(\mathbb{C}), \mathbb{Z}/2)$.*

Note that, by Theorem 1.4, Conjecture 1.6 implies Conjecture 1.7. Providing evidence for Conjecture 1.6 requires the development of techniques for computing the topological space $|\mathcal{X}(\mathbb{R})|$; such techniques are currently unavailable, as the topology of real algebraic stacks has so far been an unexplored area. These methods will be developed in the second part of this two-paper series, see [AGF25].

In the present work, we restrict ourselves to verifying Conjecture 1.6 in the case of the classifying groupoid $B\Gamma$ attached to a finite group Γ equipped with an involution. In this setting, Conjecture 1.6 is equivalent to a group-theoretic statement, whose proof has been kindly communicated to us by Will Sawin, cf. [Wil25]. The result is as follows.

Proposition 1.8. *Let Γ be a finite group and $\sigma\colon \Gamma \to \Gamma$ an involution. Define $B\Gamma = [\Gamma \rightrightarrows \mathrm{pt}]$, endowed with the involution induced by σ. The inequality (3) holds for $B\Gamma$.*

1.5 Variant for groupoid cohomology. Our construction of the fixed locus $|\mathscr{X}^G|$ of a topological groupoid with involution $\mathscr{X}$ actually proceeds by first associating to $\mathscr{X}$ a topological groupoid of fixed points $\mathscr{X}^G$, see Definition 3.4; the space $|\mathscr{X}^G|$ is then defined as the coarse space associated to this fixed-point groupoid. From this perspective, Conjecture 1.6 compares the topology of the coarse space $|\mathscr{X}^G|$ of $\mathscr{X}^G$ with the topology of the coarse space $|\mathcal{I}_{\mathscr{X}}|$ of the inertia groupoid of $\mathscr{X}$. In a complementary direction, one could compare the *groupoid cohomology* of $\mathscr{X}^G$ with the groupoid cohomology of $\mathscr{X}$, as an alternative route towards generalizing the Smith–Thom inequality (2) to topological groupoids. However, this framework makes it more difficult to formulate a precise conjecture, since groupoid cohomology is often infinite-dimensional (think of the case $\mathscr{X} = B\Gamma$ for $\Gamma = \mathbb{Z}/2$). We conclude the paper with Section 7, where we explore possible variants of Conjecture 1.6 in the context of groupoid cohomology, presenting several related questions and examples.

1.6 Organization of the paper. The paper is organized as follows. In Section 2, we collect some notation and conventions. In Section 3, we briefly review the theory of topological groupoids, define the fixed groupoid of a topological G-groupoid, and prove some preliminary results. Section 4.3 is devoted to the proof of Theorem 1.4, and Section 5 to the proof of Theorem 1.5. In Section 6, we prove Proposition 1.8. We conclude the paper in Section 7 with some speculative remarks on a possible analogue of the Smith–Thom inequality in the context of groupoid cohomology.

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2 Notation and conventions

We let G denote the finite group $G \coloneqq \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$, and $\sigma \in G$ a generator. A topological groupoid (U, R, s, t, c, e, i), consisting of topological spaces U and R and continuous maps $s, t \colon R \to U$, $c \colon R \times_{s,U,t} R \to R$, $e \colon U \to R$ and $i \colon R \to R$ that satisfy the usual compatibility conditions (see [Stacks, Tag 0230]), will be denoted by $[R \rightrightarrows U]$.

A *variety* over $\mathbb{R}$ (resp. $\mathbb{C}$) will be a reduced and separated scheme of finite type over $\mathbb{R}$ (resp. $\mathbb{C}$). For a scheme S, and $(\mathsf{Sch}/S)_{fppf}$ a big fppf site of S (cf. [Stacks, Tag 021S]), an *algebraic stack* over S is a stack in groupoids $p \colon \mathcal{X} \to (\mathsf{Sch}/S)_{fppf}$

(see [Stacks, Tag 0304]) which satisfies the conditions in [Stacks, Tag 026O]. We will indicate an algebraic stack by a calligraphic letter, such as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$. Schemes are usually indicated by roman capitals, such as X, Y, Z. For an algebraic stack $\mathcal{X}$, we let $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ denote the inertia stack over $\mathcal{X}$. When $\mathcal{X}$ is an algebraic stack over a scheme S, we let $|\mathcal{X}(S)|$ denote the set of isomorphism classes of the groupoid $\mathcal{X}(S)$.

An algebraic stack $\mathcal{X}$ over S is a *Deligne–Mumford stack* if there exists a scheme U and a surjective étale morphism $U \to \mathcal{X}$. We will repeatedly use the following theorem by Keel and Mori [KM97]: if $\mathcal{X}$ is a separated Deligne–Mumford stack locally of finite type over a scheme S, there exists a coarse moduli space $\mathcal{X} \to M$. In particular, if $S = \mathrm{Spec}(\mathbb{R})$, then M is an algebraic space locally of finite type over $\mathbb{R}$, and hence its complex locus $M(\mathbb{C})$ has naturally the structure of a complex-analytic space (see [Knu71, Chapter I, Proposition 5.18]) endowed with an anti-holomorphic involution with fixed locus $M(\mathbb{R})$. This provides the real locus $M(\mathbb{R})$ of M with the structure of a closed real analytic subspace $M(\mathbb{R}) \subset M(\mathbb{C})$ and in particular with a topology.

3 Topological groupoids with involution

In this section, we prove some preliminary results on topological groupoids with involution. These results will be used in Sections 4 and 5 to study the real locus $|\mathcal{X}(\mathbb{R})|$ of a Deligne–Mumford stack $\mathcal{X}$ over $\mathbb{R}$.

3.1 Topological groupoids. Recall that a topological groupoid is a groupoid object in the category of topological spaces (see e.g. [CM00, Section 1]). As such, a topological groupoid $\mathscr{X} = [R \rightrightarrows U]$ consist of two topological spaces, U (the space of objects) and R (the space of arrows), and a collection of continuous maps $s\colon R \to U$ (source), $t\colon R \to U$ (target), $c\colon R \times_U R \to R$ (composition), $e\colon U \to R$ (identity) and $i\colon R \to R$ (inversion), satisfying various natural compatibilities (see e.g. [Stacks, Tag 0230]).

Definition 3.1. One defines $|\mathscr{X}| = U/R = U/_{\cong}$, the set of isomorphism classes of objects $x \in U$, and one equips $|\mathscr{X}|$ with the quotient topology induced by $U \to |\mathscr{X}|$.

For later use, we make the following remark.

Lemma 3.2. *Assume that $t\colon R \to U$ is open. Then the quotient map $\pi\colon U \to |\mathscr{X}| = U/R$ is open.*

Proof. For $B \subset U$ open, we have that $\pi^{-1}(\pi(B)) = t(s^{-1}(B))$ is open in U. Hence $\pi(B)$ is open in U/R. □

In the classical Smith-Thom inequality (2), one assumes that the topological spaces are locally compact Hausdorff; we give a criterion to guarantee that the coarse space associated to a topological groupoid is locally compact Hausdorff.

Lemma 3.3. *Let $\mathscr{X} = [R \rightrightarrows U]$ be a topological groupoid. Assume that U and R are locally compact Hausdorff. Suppose the quotient map $\pi \colon U \to |\mathscr{X}| = U/R$ is open and the map $(s,t)\colon R \to U \times U$ is closed. Then $|\mathscr{X}| = U/R$ is locally compact Hausdorff.*

Proof. Let $Z \subset U \times U$ be the image of $(s,t)\colon R \to U \times U$. Let $\sim$ be the equivalence relation on U such that $x \sim y$ if and only if $x = y$ or $(x,y) \in Z$. Since U is Hausdorff, $\pi\colon U \to U/_\sim = U/R$ is open and $Z \subset U \times U$ is closed, we have that $|\mathscr{X}| = U/_\sim$ is Hausdorff. Since π is open and U is locally compact, $|\mathscr{X}|$ is locally compact. □

3.2 Topological groupoids with involution. Recall that if $\mathscr{X}_1 = [R_1 \rightrightarrows U_1]$ and $\mathscr{X}_2 = [R_2 \rightrightarrows U_2]$ are two topological groupoids, then a morphism $F\colon \mathscr{X}_1 \to \mathscr{X}_2$ is a pair (f,g) of continuous maps $f\colon R_1 \to R_2$ and $g\colon U_1 \to U_2$ compatible with all the structure maps of the groupoids. A *topological G-groupoid* (or a *topological groupoid with involution*) is a pair $(\mathscr{X}, \sigma)$, where $\mathscr{X}$ is a topological groupoid and $\sigma\colon \mathscr{X} \to \mathscr{X}$ is an involution (i.e., a morphism that satisfies $\sigma^2 = \sigma$).

Let $\mathscr{X} = [R \rightrightarrows U]$ be a topological G-groupoid. We now define the appropriate analogue of the fixed locus of the involution.

Definition 3.4. The *groupoid of fixed points* is the topological groupoid $\mathscr{X}^G \coloneqq [\mathrm{A}^1(G,R) \rightrightarrows \mathrm{Z}^1(G,R)]$, where:

- $\mathrm{Z}^1(G,R)$ is the subspace $\mathrm{Z}^1(G,R) \subseteq U \times R$ of pairs (x,φ) such that φ is an isomorphism $x \xrightarrow{\sim} \sigma(x)$ such that $\sigma(\varphi)\circ\varphi = \mathrm{id}$; in particular we have two natural maps $\mathrm{Z}^1(G,R) \to U, (x,\varphi) \mapsto x$ and $\mathrm{Z}^1(G,R) \to R, (x,\varphi) \mapsto \varphi$.
- $\mathrm{A}^1(G,R)$ is the fibre product $\mathrm{A}^1(G,R) = R \times_{s,U} \mathrm{Z}^1(G,R)$; more explicitly an arrow $f\colon (x,\varphi) \to (x',\varphi')$ between two objects $(x,\varphi),(x',\varphi') \in \mathrm{Z}^1(G,R)$ is given by an arrow $f \in R$ with $s(f) = x$ and $t(f) = x'$, such that $\sigma(f)\circ\varphi = \varphi'\circ f$ as maps $x \to \sigma(x')$.
- The maps $s,t\colon \mathrm{A}^1(G,R) \to \mathrm{Z}^1(G,R)$ are defined as $s(f,(x,\varphi)) = (x,\varphi) = (s(f),\varphi)$ and $t(f,(x,\varphi)) = (t(x), \sigma(f)\circ\varphi\circ f^{-1})$.
- The inversion map $i\colon \mathrm{A}^1(G,R) \to \mathrm{A}^1(G,R)$ is defined by sending an arrow $f\colon (x,\varphi) \to (x',\varphi')$ with $f \in R$ to the arrow $f^{-1} \in R$. The composition of two arrows is induced by the composition in R, and finally, the identity $e\colon \mathrm{Z}^1(G,R) \to \mathrm{A}^1(G,R)$ is defined by sending (x,φ) to the identity $\mathrm{id}_x\colon (x,\varphi) \to (x,\varphi)$, where $\mathrm{id}_x \in R$ is the identity arrow of $x \in U$.

With these arrows (s,t,c,e,i), one can check that $\mathscr{X}^G = [\mathrm{A}^1(G,R) \rightrightarrows \mathrm{Z}^1(G,R)]$ is indeed a topological groupoid.

Example 3.5. Let $\mathscr{T}$ be a stack on the site of all topological spaces. Define an *involution* on $\mathscr{T}$ to be a 1-morphism $\sigma\colon \mathscr{T} \to \mathscr{T}$ such that σ^2 is 2-isomorphic to the identity functor (where 2-isomorphic should be taken in the sense of [Stacks, Tag 02ZK]). Say that a *topological G-stack* consists of a topological stack $\mathscr{T}$ in the sense of [Noo12, Section 2] and an involution σ on $\mathscr{T}$, such that there exists a representable surjective morphism $p\colon U \to \mathscr{T}$ where U is a topological space equipped with an involution $\sigma_U\colon U \to U$ that 2-commutes with σ and p. For such a stack $\mathscr{T}$ with presentation $p\colon U \to \mathscr{T}$, let R be a topological space with $R \cong U \times_{\mathscr{T}} U$. Then R carries a natural involution $\sigma_R\colon R \to R$ induced by σ and σ_U, and the involutions σ_U and σ_R induce on $\mathscr{X} = [R \rightrightarrows U]$ the structure of a topological G-groupoid.

Example 3.6. If Γ is a finite group equipped with an involution $\sigma\colon \Gamma \to \Gamma$, and if $\mathscr{X} = B\Gamma = [\Gamma \rightrightarrows \mathrm{pt}]$ is the classifying groupoid of Γ, then there is a natural isomorphism of topological groupoids

$$\mathscr{X}^G = \coprod_{[\gamma]\in \mathrm{H}^1(G,\Gamma)} B(\Gamma^{\sigma_\gamma})$$

where, for an element $\gamma \in \Gamma$ with $\gamma\sigma(\gamma) = e$, the subgroup $\Gamma^{\sigma_\gamma} \subseteq \Gamma$ is made by the $g \in \Gamma$ such that $\gamma\sigma(g) = g\gamma$.

3.3 Properties of the groupoid of fixed points. In order to prove Theorem 1.4, we need Proposition 3.10 below, which is the main result of this section. This proposition roughly says that the formation of $\left|\mathscr{X}^G\right|$ is invariant under a base change $U' \to U$ with suitable properties. To prove it, we need a definition and two lemmas.

Definition 3.7. (Compare [CM00, Section 1.5].) Let $\mathscr{X}_1 = [R_1 \rightrightarrows U_1]$ and $\mathscr{X}_2 = [R_2 \rightrightarrows U_2]$ be topological groupoids, and let $F\colon \mathscr{X}_2 \to \mathscr{X}_1$ be a morphism of topological groupoids, defined by a pair of compatible maps $(f\colon U_2 \to U_1, g\colon R_2 \to R_1)$. We say that F is an *open equivalence* if the following conditions hold: the map $f\colon U_2 \to U_1$ is open, the composition

$$R_1 \times_{s,U_1} U_2 \to R_1 \xrightarrow{t} U_1 \tag{4}$$

is open and surjective, and the diagram

$$\begin{array}{ccc} R_2 & \xrightarrow{\quad g \quad} & R_1 \\ \Big\downarrow{\scriptstyle (s,t)} & & \Big\downarrow{\scriptstyle (s,t)} \\ U_2 \times U_2 & \xrightarrow{f \times f} & U_1 \times U_1 \end{array} \tag{5}$$

is cartesian.

Lemma 3.8. *Let $F\colon \mathscr{X}_2 \to \mathscr{X}_1$ be a morphism of topological groupoids $\mathscr{X}_i = [R_i \rightrightarrows U_i]$ for which $s, t\colon R_i \to U_i$ are surjective and open ($i = 1, 2$), and assume that F is an open equivalence. Then the induced map $|F| : |\mathscr{X}_2| \to |\mathscr{X}_1|$ is a homeomorphism.*

Proof. Note first that $|F|$ is bijective. Indeed, the surjectivity of (4) implies that each $x \in U_1$ is isomorphic to an object in the image of $f\colon U_2 \to U_1$, hence $|F|$ is surjective; the injectivity of $|F|$ follows from from the fact that diagram (5) is cartesian.

The map $|F|$ is continuous (since $|F|$ lifts to a continuous map $U_2 \to U_1$) hence it remains to prove that $|F|$ is open. This holds in view of the commutative diagram

$$\begin{array}{ccc} U_1 & \xrightarrow{\quad f \quad} & U_2 \\ \Big\downarrow & & \Big\downarrow \\ U_1/R_1 & \xrightarrow{|F|} & U_2/R_2, \end{array}$$

since the vertical arrows are open by Lemma 3.2 and the upper horizontal arrow is open by assumption. □

Lemma 3.9. *Let $\mathscr{X} = [R \rightrightarrows U]$ be a topological G-groupoid. Assume that the maps $s, t\colon R \to U$ are surjective and open. Then the maps $s, t\colon \mathrm{A}^1(G, R) \to \mathrm{Z}^1(G, R)$ are surjective and open. Similarly, if $s, t\colon R \to U$ are local homeomorphisms, the same holds for $s, t\colon \mathrm{A}^1(G, R) \to \mathrm{Z}^1(G, R)$.*

Proof. Since the inversion $i\colon \mathrm{A}^1(G, R) \xrightarrow{\sim} \mathrm{A}^1(G, R)$ is a homeomorphism, and since $s \circ i = t$ as maps $\mathrm{A}^1(G, R) \to \mathrm{Z}^1(G, R)$, the assertions for t follow from the assertions for s. The assertions for s follow from the fact that the following diagram is cartesian:

$$\begin{array}{ccc} \mathrm{A}^1(G, R) & \xrightarrow{\quad s \quad} & \mathrm{Z}^1(G, R) \\ \Big\downarrow & & \Big\downarrow \\ R & \xrightarrow{\quad s \quad} & U. \end{array}$$

This ends the proof. □

Proposition 3.10. *Let $\mathscr{X}_1 = [R_1 \rightrightarrows U_1]$ and $\mathscr{X}_2 = [R_2 \rightrightarrows U_2]$ be topological G-groupoids, such that the maps $s, t\colon R_i \to U_i$ are surjective and open for $i = 1, 2$. Let $F\colon \mathscr{X}_2 \to \mathscr{X}_1$ be an open equivalence. Then the map*

$$F^G\colon \mathscr{X}_2^G \to \mathscr{X}_1^G$$

is an open equivalence. In particular, the map $\left|F^G\right| : \left|\mathscr{X}_2^G\right| \to \left|\mathscr{X}_1^G\right|$ is a homeomorphism.

Proof. Let $f\colon U_2 \to U_1$ and $g\colon R_2 \to R_1$ be the maps corresponding to F. By Lemma 3.8, it suffices to prove that the map $F^G\colon \mathscr{X}_2^G \to \mathscr{X}_1^G$ is an open equivalence. Note that $s, t\colon \mathrm{A}^1(G, R_i) \to \mathrm{Z}^1(G, R_i)$ are surjective and open by Lemma 3.9. By construction, the map $\mathrm{Z}^1(G, R_i) \to R_i$ defined as $(x, \varphi) \mapsto \varphi$ is an embedding. For $i = 1, 2$, consider the restriction $(s, t)\colon \mathrm{Z}^1(G, R_i) \to U_i \times U_i$ of the map $(s, t)\colon R_i \to U_i \times U_i$ to $\mathrm{Z}^1(G, R_i)$. Observe that the diagram

$$\begin{array}{ccc}
\mathrm{Z}^1(G, R_2) & \xrightarrow{(s,t)} & U_2 \times U_2 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\times f} \\
\mathrm{Z}^1(G, R_1) & \xrightarrow{(s,t)} & U_1 \times U_1
\end{array}$$

is cartesian. As $U_2 \times U_2 \to U_1 \times U_1$ is open, it follows that the map $\mathrm{Z}^1(G, R_2) \to \mathrm{Z}^1(G, R_1)$ is open.

Next, observe that the squares in the following diagram are cartesian:

$$\begin{array}{ccccc}
\mathrm{A}^1(G, R_1) \times_{s, \mathrm{Z}^1(G,R_1)} \mathrm{Z}^1(G, R_2) & \longrightarrow & \mathrm{A}^1(G, R_1) & \xrightarrow{t} & \mathrm{Z}^1(G, R_1) \\
\downarrow & & \downarrow & & \downarrow \\
R_1 \times_{s, U_1} U_2 & \longrightarrow & R_1 & \xrightarrow{t} & U_1.
\end{array}$$

Consequently, since the bottom row is surjective, the top row is surjective as well.

It remains to show that the diagram

$$\begin{array}{ccc}
\mathrm{A}^1(G, R_2) & \xrightarrow{g} & \mathrm{A}^1(G, R_1) \\
\downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\
\mathrm{Z}^1(G, R_2) \times \mathrm{Z}^1(G, R_2) & \xrightarrow{f\times f} & \mathrm{Z}^1(G, R_1) \times \mathrm{Z}^1(G, R_1)
\end{array}$$

is cartesian. This follows from the fact that (5) is cartesian and the proof is concluded. □

4 Topological groupoids and Deligne–Mumford stacks

The goal of this section is prove Theorem 1.4. In order to do that, in Section 4.1 we recall the definition of a topological groupoids and its relation with Deligne-Mumford stacks. In Section 4.2 we introduction the notion of topological stacks with involution and we explain their relation with real Deligne-Mumford stack. In Section 4.3 we recall the definition of real analytic topology on a real Deligne-Mumford stack and we state Theorem 4.8, which is a reformulation of Theorem 1.4 in the introduction. We then use the preliminaries of Section 3 to prove Theorem 4.8.

4.1 Topological groupoids arising from complex Deligne-Mumford stacks. For a Deligne–Mumford stack $\mathcal{X}$ locally of finite type over $\mathbb{C}$, we view the set of isomorphism classes $|\mathcal{X}(\mathbb{C})|$ of the groupoid $\mathcal{X}(\mathbb{C})$ as a topological space, by equipping it with the quotient topology induced by the surjective morphism $U(\mathbb{C}) \to |\mathcal{X}(\mathbb{C})|$, where U is a scheme and $U \to \mathcal{X}$ a surjective étale morphism. This topology on $|\mathcal{X}(\mathbb{C})|$ does not depend on the choice of étale presentation $U \to \mathcal{X}$. Every complex Deligne–Mumford stack gives rise to a topological groupoid in the following way.

Construction 4.1. Let $\mathcal{X}$ be a complex Deligne–Mumford stack, so that there exists an étale surjective presentation $\pi\colon U \to \mathcal{X}$ by a scheme. Let R be a scheme with $R \cong U \times_{\mathcal{X}} U$, so that the two projection maps $U \times_{\mathcal{X}} U \to U$ yield two maps $R \to U$ that turn $[R \rightrightarrows U]$ into a groupoid scheme. Then $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ is a topological groupoid.

Via the bijection $R(\mathbb{C}) \cong (U \times_{\mathcal{X}} U)(\mathbb{C})$, any $r \in R(\mathbb{C})$ corresponds to an element $(x, y, \alpha) \in (U \times_{\mathcal{X}} U)(\mathbb{C})$ consisting of two elements $x, y \in U(\mathbb{C})$ and an isomorphism $\alpha\colon \pi(x) \xrightarrow{\sim} \pi(y)$. This yields a functor $F\colon \mathscr{X} \to \mathcal{X}(\mathbb{C})$ that sends an object $x \in U(\mathbb{C})$ to the object $\pi(x) \in \mathcal{X}(\mathbb{C})$ and an arrow $r = (x, y, \alpha) \in R(\mathbb{C})$ to the isomorphism $\alpha\colon \pi(x) \xrightarrow{\sim} \pi(y)$. The functor $F\colon \mathscr{X} \to \mathcal{X}(\mathbb{C})$ is an equivalence of categories.

Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type over $\mathbb{C}$. If $\mathcal{X}$ is separated, then it has a coarse moduli space $\mathcal{X} \to M$ by [KM97]. By Construction 4.1 and Lemma 3.2, it follows that the map $|\mathcal{X}(\mathbb{C})| \to M(\mathbb{C})$ is a homeomorphism (this can also been seen by using that $\mathcal{X}$ is étale locally over M a finite quotient stack, see [AV02, Lemma 2.2.3]).

4.2 Topological groupoids arising from real Deligne–Mumford stacks. We will often make use of the following definition.

Definition 4.2. A *real DM stack* is a separated Deligne–Mumford stack of finite type over $\mathbb{R}$.

Every real DM stack $\mathcal{X}$ give rise to a topological G-groupoid $(\mathscr{X}, \sigma)$ in the following way.

Construction 4.3. Let $\mathcal{X}$ be a real DM stack, and choose a scheme U over $\mathbb{R}$ and an étale surjective morphism $U \to \mathcal{X}$. Let R be a scheme with $R \cong U \times_{\mathcal{X}} U$, so that we get a groupoid scheme $[R \rightrightarrows U]$, see Construction 4.1. Since U and R are schemes locally of finite type over $\mathbb{R}$, $U(\mathbb{C})$ and $R(\mathbb{C})$ admit natural anti-holomorphic involutions $\sigma\colon U(\mathbb{C}) \to U(\mathbb{C})$ and $\sigma\colon R(\mathbb{C}) \to R(\mathbb{C})$, compatible with the structure maps of the groupoid. Hence $[R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ is a topological groupoid with involution.

Let $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ be the topological groupoid with involution associated to a real Deligne-Mumford stack $\mathcal{X} = [U/R]$ as in Construction 4.3. We aim to construct a natural equivalence of categories $F\colon \mathscr{X}^G \to \mathcal{X}(\mathbb{R})$.

To do this, recall that any $r \in R(\mathbb{C}) = (U \times_{\mathcal{X}} U)(\mathbb{C})$ corresponds to a triple $r = (x, y, \alpha)$ with $x, y \in U(\mathbb{C})$ and $\alpha\colon \pi(x) \xrightarrow{\sim} \pi(y)$ an isomorphism, where π is the map $U \to \mathcal{X}$. Thus, we have

$$\mathrm{Z}^1(G, R(\mathbb{C})) = \{\omega = (x, (x, \sigma(x), \varphi)) \in U(\mathbb{C}) \times R(\mathbb{C}) \mid \sigma(\varphi) \circ \varphi = \mathrm{id}\}.$$

For $\omega = (x, (x, \sigma(x), \varphi)) \in \mathrm{Z}^1(G, R(\mathbb{C}))$, we get an element $\pi(x) \in \mathcal{X}(\mathbb{C})$ and an isomorphism $\varphi\colon \pi(x) \xrightarrow{\sim} \pi(\sigma(x))$ such that $\sigma(\varphi) \circ \varphi = \mathrm{id}$. By Galois descent (cf. [Gro60]), this yields an object $F(\omega) \in \mathcal{X}(\mathbb{R})$. Similarly, any arrow $f\colon \omega \to \omega'$, $f \in \mathrm{A}^1(G, R(\mathbb{C}))$, is given by an arrow $f = (x, x', \alpha) \in R(\mathbb{C})$ such that $\varphi' \circ \alpha = \sigma(\alpha) \circ \varphi$ as maps $\pi(x) \to \pi(\sigma(x'))$, and this yields an arrow $F(f)\colon F(\omega) \to F(\omega')$ in $\mathcal{X}(\mathbb{R})$, again by Galois descent. This gives a natural functor

$$F\colon \mathscr{X}^G \to \mathcal{X}(\mathbb{R}).$$

Lemma 4.4. *The functor $F\colon \mathscr{X}^G \to \mathcal{X}(\mathbb{R})$ is an equivalence of categories. In particular, we get a bijection $|F| : \left|\mathscr{X}^G\right| \xrightarrow{\sim} |\mathcal{X}(\mathbb{R})|$.*

Proof. This follows from the theory of descent, see [Gro60]. □

Proposition 4.5. *Let $\mathcal{X}$ be a real DM stack. Let U be a scheme with a surjective étale morphism $U \to \mathcal{X}$, consider the resulting topological G-groupoid $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ (cf. Construction 4.3). Then the topology on $|\mathcal{X}(\mathbb{R})|$ induced by the topology of $\left|\mathscr{X}^G\right|$ and the bijection $|F| : \left|\mathscr{X}^G\right| \xrightarrow{\sim} |\mathcal{X}(\mathbb{R})|$ of Lemma 4.4 does not depend on the choice of the surjective étale presentation $U \to \mathcal{X}$.*

Proof. Let $U_i \to \mathcal{X}$ $(i = 1, 2)$ be two surjective étale morphisms with U_i schemes, and let R_i be schemes with $R_i \cong U_i \times_{\mathcal{X}} U_i$. Let U_3 be a scheme with an isomorphism $U_3 \cong U_1 \times_{\mathcal{X}} U_2$. Let R_3 be a scheme with an isomorphism $R_3 \cong U_3 \times_{\mathcal{X}} U_3$. By Construction 4.3, this defines three topological G-groupoids

$$\mathscr{X}_i := [R_i(\mathbb{C}) \rightrightarrows U_i(\mathbb{C})], \qquad i = 1, 2, 3.$$

Then for $i = 1, 2$ we obtain a G-equivariant morphism of topological G-groupoids $\Phi_i \colon \mathscr{X}_3 \to \mathscr{X}_i$. We claim that for $i = 1, 2$, the morphism Φ_i is an open equivalence (in the sense of Definition 3.7). To prove this, note that $R_3 \cong (U_3 \times_{\mathbb{R}} U_3) \times_{(U_1 \times_{\mathbb{R}} U_1)} R_1$ and $R_3 \cong (U_3 \times_{\mathbb{R}} U_3) \times_{(U_2 \times_{\mathbb{R}} U_2)} R_2$. Moreover, since $U_3 \to U_i$ is étale for $i = 1, 2$, the resulting maps $U_3(\mathbb{C}) \to U_1(\mathbb{C})$ and $U_3(\mathbb{C}) \to U_2(\mathbb{C})$ are local homeomorphisms (see [Gro71, Exposé XII, Proposition 3.1 & Remarque 3.3]). In particular, the maps $U_3(\mathbb{C}) \to U_i(\mathbb{C})$ are open for $i = 1, 2$. As the map $U_3(\mathbb{C}) \to U_i(\mathbb{C})$ is surjective for $i = 1, 2$, the composition $R_i(\mathbb{C}) \times_{s, U_i(\mathbb{C})} U_3(\mathbb{C}) \to R_i(\mathbb{C}) \xrightarrow{t} U_i(\mathbb{C})$ is surjective for $i = 1, 2$, proving the claim.

Consequently, by Proposition 3.10, the induced maps

$$|\mathscr{X}_1^G| \leftarrow |\mathscr{X}_3^G| \to |\mathscr{X}_2^G|$$

are homeomorphisms. This proves the proposition. ☐

4.3 The real analytic topology of a real DM stack. For a real DM stack $\mathcal{X}$, the set of isomorphism classes $|\mathcal{X}(\mathbb{R})|$ of its real locus $\mathcal{X}(\mathbb{R})$ has a natural topology, generalizing the euclidean topology on $X(\mathbb{R})$ when X is a scheme. Indeed, we have the following theorem.

Theorem 4.6. *Let $\mathcal{X}$ be a real DM stack. There exists a scheme U over $\mathbb{R}$ and a surjective étale morphism $U \to \mathcal{X}$ such that $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective.*

Proof. See [GF22a, Theorem 2.9] or [GF22b, Theorem 7.4]. ☐

Definition 4.7. (cf. [GF22b, Definition 7.5]) Let $\mathcal{X}$ be a real DM stack. The *real analytic topology* on $|\mathcal{X}(\mathbb{R})|$ is defined as follows. Choose a scheme U over $\mathbb{R}$ and a surjective étale morphism $U \to \mathcal{X}$ such that $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective. Then consider the real analytic topology on $U(\mathbb{R})$, and give $|\mathcal{X}(\mathbb{R})|$ the quotient topology induced by the surjection $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$.

One shows that the real analytic topology is independent of the choice of an étale presentation that is essentially surjective on real points, see [GF22b, Proposition 7.6].

Let now $\mathcal{X}$ be a real DM stack and fix a surjective étale morphism $U \to \mathcal{X}$. Let $\mathscr{X} = [R(\mathbb{C}) \rightrightarrows U(\mathbb{C})]$ be the topological groupoid with involution associated to a real Deligne-Mumford stack $\mathcal{X} = [U/R]$ as in Construction 4.3.

Theorem 4.8. *Consider the groupoid of fixed points $\mathscr{X}^G$, see Definition 3.4, with attached topological space $\left|\mathscr{X}^G\right|$. Consider $|\mathcal{X}(\mathbb{R})|$ as a topological space via the real analytic topology. The bijection $|F| : \left|\mathscr{X}^G\right| \cong |\mathcal{X}(\mathbb{R})|$ of Lemma 4.4 is a homeomorphism.*

Proof. By Proposition 4.5, to prove the theorem, we may replace our surjective étale morphism $U \to \mathcal{X}$ by any other surjective étale morphism $U' \to \mathcal{X}$ with U' a scheme. In particular, by Theorem 4.6, we may assume that $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective. By definition of the real analytic topology on $|\mathcal{X}(\mathbb{R})|$, it therefore suffices to show that the topology of $\left|\mathscr{X}^G\right|$ agrees with the quotient topology induced by the surjection $U(\mathbb{R}) \to \left|\mathscr{X}^G\right|$ and the real analytic topology of $U(\mathbb{R})$. This holds, because the map $U(\mathbb{R}) \to \left|\mathscr{X}^G\right|$ is open by Lemmas 3.2 and 3.9. We are done. □

5 Topology of real gerbes

In this section, we prove Theorem 1.5. To that end, in Section 5.1, we begin by studying the topology of the complex inertia. Section 5.2 is devoted to basic properties of the maps induced on complex points by the inertia morphism and of the maps on real points induced by the natural morphism from a separated Deligne–Mumford stack to its coarse moduli space. In Section 5.3, we study the stack $[U/H]$ associated to a finite étale group scheme $H \to U$ over a real variety U. Finally, in Section 5.4, we combine these ingredients to prove Theorem 1.5.

5.1 Preliminaries on the complex inertia. A central fact for the theory developed in this paper and its sequel [AGF25] is the following: for a separated complex Deligne–Mumford stack $\mathcal{X}$, with coarse moduli space $\mathcal{X} \to M$ and coarse moduli space of the inertia stack $\mathcal{I}_{\mathcal{X}} \to I_{\mathcal{X}}$, the induced morphism of complex analytic spaces $I_{\mathcal{X}}(\mathbb{C}) \to M(\mathbb{C})$ is closed with finite fibers. We establish this result in Lemma 5.2 below. A key ingredient in the proof is the following technical lemma, which is likely well-known, but for which we include a proof due to the lack of a suitable reference.

Lemma 5.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be separated Deligne–Mumford stacks locally of finite type over $\mathbb{C}$, with coarse moduli spaces $\mathcal{X} \to M_{\mathcal{X}}$ and $\mathcal{Y} \to M_{\mathcal{Y}}$. For a finite morphism of stacks $\mathcal{X} \to \mathcal{Y}$, the induced morphism of coarse moduli spaces $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ is finite.*

Proof. First, we claim that the morphism on coarse moduli spaces $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ induced by $\mathcal{X} \to \mathcal{Y}$ is separated. Since $M_{\mathcal{X}}$ and $M_{\mathcal{Y}}$ are coarse moduli spaces of separated Deligne–Mumford stacks over $\mathbb{C}$, they are both separated over $\mathbb{C}$ (cf. [Con05, Theorem 1.1]). Consider the factorization $M_{\mathcal{X}} \xrightarrow{\alpha} M_{\mathcal{X}} \times_{M_{\mathcal{Y}}} M_{\mathcal{X}} \xrightarrow{\beta} M_{\mathcal{X}} \times_{\mathbb{C}} M_{\mathcal{X}}$ of the diagonal $\Delta = \beta \circ \alpha \colon M_{\mathcal{X}} \to M_{\mathcal{X}} \times_{\mathbb{C}} M_{\mathcal{X}}$. The map β, which is the base change of $M_{\mathcal{Y}} \to M_{\mathcal{Y}} \times_{\mathbb{C}} M_{\mathcal{Y}}$ along the map $M_{\mathcal{X}} \times_{\mathbb{C}} M_{\mathcal{X}} \to M_{\mathcal{Y}} \times_{\mathbb{C}} M_{\mathcal{Y}}$, is a closed immersion since $M_{\mathcal{Y}}$ is separated. As $M_{\mathcal{X}}$ is separated, $\Delta = \beta \circ \alpha$ is a closed immersion. Hence α is a closed immersion by [Stacks, Tag 0AGC], so that $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ is separated, as wanted.

Next, pick a finite surjective morphism $Y \to \mathcal{Y}$ where Y is a scheme; such a morphism exists by [LMB00, Theorem 16.6, page 153]. We claim that the composition $Y \to \mathcal{Y} \to M_{\mathcal{Y}}$ is finite. To see this, note that $Y \to M_{\mathcal{Y}}$ is proper and quasi-finite because $\mathcal{Y} \to M_{\mathcal{Y}}$ is proper and quasi-finite (see [Con05, Theorem 1.1]) and $Y \to \mathcal{Y}$ is finite. Therefore, $Y \to M_{\mathcal{Y}}$ is finite, see [LMB00, Corollaire (A.2.1), page 198].

Define $X = Y \times_{\mathcal{Y}} \mathcal{X}$. Since $\mathcal{X} \to \mathcal{Y}$ is finite, the base change $X \to Y$ is finite. As $Y \to M_{\mathcal{Y}}$ is finite by the above, we conclude that the composition $X \to Y \to M_{\mathcal{Y}}$ is finite. This composition agrees with the composition $X \to M_{\mathcal{X}} \to M_{\mathcal{Y}}$, in which $X \to M_{\mathcal{X}}$ is surjective and $X \to M_{\mathcal{Y}}$ is finite. Hence $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ is locally quasi-finite by [Stacks, Tag 03MJ] and [Stacks, Tag 0GWS]. Since $X \to M_{\mathcal{X}}$ is surjective and $X \to M_{\mathcal{Y}}$ quasi-compact, the morphism $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ is quasi-compact (see [Stacks, Tag 040W]). We conclude that $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ is quasi-finite, and in particular of finite type.

We claim that $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ is proper. This follows from [Stacks, Tag 08AJ] because $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ is separated and of finite type, $X \to M_{\mathcal{X}}$ is surjective, and $X \to M_{\mathcal{Y}}$ is proper (being the composition of the proper morphisms $X \to Y$ and $Y \to M_{\mathcal{Y}}$).

We conclude that $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ proper and quasi-finite. By [LMB00, Corollaire (A.2.1), page 198], this implies that $M_{\mathcal{X}} \to M_{\mathcal{Y}}$ is finite. □

Lemma 5.2. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack locally of finite type over $\mathbb{C}$. Let $\mathcal{I}_{\mathcal{X}} \to I_{\mathcal{X}}$ and $\mathcal{X} \to M$ be the coarse moduli spaces of $\mathcal{I}_{\mathcal{X}}$ and $\mathcal{X}$.*

1. *The morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is finite.*
2. *The morphism $I_{\mathcal{X}} \to M$ is finite and surjective.*
3. *The morphism of complex analytic spaces $I_{\mathcal{X}}(\mathbb{C}) \to M(\mathbb{C})$ is surjective and closed with finite fibers.*

Proof. The diagonal morphism $\Delta \colon \mathcal{X} \to \mathcal{X} \times_{\mathbb{C}} \mathcal{X}$ is of finite type, see [LMB00, Lemme (4.2), page 26]. Therefore, for each scheme S over $\mathbb{C}$ and each $x \in \mathcal{X}(S)$, the automorphism group scheme $\underline{\mathrm{Aut}}_S(x)$ of x over S is of finite type over S. Moreover, Δ is quasi-finite by [LMB00, Lemme (4.2)]. In particular, $\underline{\mathrm{Aut}}_S(x)$ is finite over S. This proves item 1. As $I_{\mathcal{X}} \to M$ is surjective, item 2 follows from item 1 and Lemma 5.1. Finally, if a map of analytic spaces $f_{\mathbb{C}}\colon X(\mathbb{C}) \to Y(\mathbb{C})$ is induced by a finite surjective morphism $f\colon X \to Y$ of algebraic spaces which are locally of finite type over $\mathbb{C}$, then $f_{\mathbb{C}}$ is surjective and closed with finite fibers. Item 3 follows thus from item 2. ☐

For an algebraic stack $\mathcal{X}$, see [Stacks, Tag 06QC] for the notions of *gerbe* and *gerbe over an algebraic stack* $\mathcal{Y}$. For example, $\mathcal{X}$ is a gerbe if there exists an algebraic space X and a morphism $\mathcal{X} \to X$ which turns $\mathcal{X}$ into a gerbe over X. See also [LMB00, (3.15) – (3.21), pages 22 – 24].

Proposition 5.3. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack locally of finite type over $\mathbb{C}$. Consider the following assertions.*

1. *The inertia $\pi\colon \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is étale over $\mathcal{X}$.*
2. *The inertia $\pi\colon \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is flat over $\mathcal{X}$.*
3. *The coarse moduli space map $\mathcal{X} \to M$ is a gerbe.*
4. *We have that $\#\mathrm{Aut}(x)$ is locally constant for $x \in \mathcal{X}(\mathbb{C})$.*

Then 1 ⇔ 2 ⇔ 3 ⇒ 4. If $\mathcal{X}$ is reduced, then all four assertions are equivalent.

Proof. We start by proving the equivalence of 2 and 3. Note that $\pi\colon \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is finite by Lemma 5.2. Therefore, by [Stacks, Tag 06QJ], the morphism $\pi\colon \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is flat if and only if $\mathcal{X}$ is a gerbe. Hence, we need to prove that $\mathcal{X}$ is a gerbe if and only if $\mathcal{X}$ is a gerbe over M. By definition, if $\mathcal{X}$ is a gerbe over M then $\mathcal{X}$ is a gerbe. Conversely, assume that $\mathcal{X}$ is a gerbe. Let $\pi_0(\mathcal{X})^{sh}$ be the sheafification of the presheaf $U \mapsto \mathrm{Ob}(\mathcal{X}(U))/_{\cong}$ on the site $(\mathsf{Sch}/\mathbb{C})_{fppf}$. By [Stacks, Tag 06QD], we have that $\pi_0(\mathcal{X})^{sh}$ is an algebraic space. On the one hand, by [LMB00, Lemme 3.18, page 23], the morphism $\mathcal{X} \to M$ factors uniquely as $\mathcal{X} \to \pi_0(\mathcal{X})^{sh} \to M$. On the other hand, since $\mathcal{X} \to M$ is the coarse moduli space of $\mathcal{X}$ and since $\pi_0(\mathcal{X})^{sh}$ is an algebraic space, the morphism $\mathcal{X} \to \pi_0(\mathcal{X})^{sh}$ factors uniquely as $\mathcal{X} \to M \to \pi_0(\mathcal{X})^{sh}$. This proves that $M \cong \pi_0(\mathcal{X})^{sh}$. Finally, by [Stacks, Tag 06QD], the natural morphism $\mathcal{X} \to \pi_0(\mathcal{X})^{sh}$ turns $\mathcal{X}$ into a gerbe over $\pi_0(\mathcal{X})^{sh} \cong M$. This proves the equivalence of assertions 2 and 3.

Trivially, assertion 1 implies assertion 2. Moreover, as a finite flat group scheme of order invertible in the base is finite étale, flatness of π implies étaleness of π. Thus, assertions 1 and 2 are equivalent, and assertion 1 readily implies assertion 4.

To conclude the proof, it is enough to show that assertion 4 implies assertion 2 if $\mathcal{X}$ is reduced. So, assume that $\mathcal{X}$ is reduced and that $\#\mathrm{Aut}(x)$ is locally constant for $x \in \mathcal{X}(\mathbb{C})$. Since $\mathcal{X}$ is reduced, there exists a reduced scheme U and a surjective étale morphism $U \to \mathcal{X}$. Let $x \in \mathcal{X}(U)$ be the object corresponding to $U \to \mathcal{X}$. We have $\mathcal{I}_{\mathcal{X}} \times_{\mathcal{X}} U = \underline{\mathrm{Aut}}_U(x) \to U$, the automorphism group scheme of x over U, and by [Stacks, Tag 04XD], it suffices to show that $\underline{\mathrm{Aut}}_U(x) \to U$ is flat. By Lemma 5.2, the morphism $\underline{\mathrm{Aut}}_U(x) \to U$ is finite, and by Cartier's theorem it has reduced fibres. As $\#\mathrm{Aut}(x)$ is locally constant for $x \in \mathcal{X}(\mathbb{C})$, it follows that the function $\rho\colon U \to \mathbb{Z}$ defined as $\rho(u) = \dim_{k(u)}(f_U^{-1}(y))$ is locally constant; since U is reduced, $\underline{\mathrm{Aut}}_U(x) \to U$ is therefore flat by [Stacks, Tag 00NX] and [Stacks, Tag 0FWG]. □

5.2 The fibers of the maps $|\mathcal{I}_{\mathcal{X}}(\mathbb{C})| \to |\mathcal{X}(\mathbb{C})|$ and $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$. For later use, and to provide further motivation for Conjecture 1.6, we show that, in the case of a stacky quotient $\mathcal{X} = [X/\Gamma]$ of a real scheme X by a finite group Γ, the fibers of the maps $\pi\colon |\mathcal{I}_{\mathcal{X}}(\mathbb{C})| \to |\mathcal{X}(\mathbb{C})| = (X/\Gamma)(\mathbb{C})$ and $f\colon |\mathcal{X}(\mathbb{R})| \to (X/\Gamma)(\mathbb{R})$ are closely related. For example, by combining Propositions 5.4 and 5.5, one sees that for a point $x \in X(\mathbb{R})$ with image $y \in (X/\Gamma)(\mathbb{R})$, we have $\pi^{-1}(y) = \Gamma_x/\Gamma_x$ and $f^{-1}(y) = \mathrm{H}^1(G, \Gamma_x)$.

5.2.1 Fibres of the coarse inertia map. Let S be a scheme of finite type over $\mathbb{C}$. Let X be a scheme and $f\colon X \to S$ a scheme of finite type over S. Let $H \to S$ be a finite group scheme over S, equipped with an action on X over S.

Define $\mathcal{X}$ as the quotient stack $[X/H]$, which we view as a stack over $\mathbb{C}$ equipped with a natural morphism $\mathcal{X} \to S$. Let $q\colon X(\mathbb{C}) \to |\mathcal{X}(\mathbb{C})|$ be the natural map of spaces over $S(\mathbb{C})$, giving the quotient map $q_s\colon X_s(\mathbb{C}) \to |\mathcal{X}(\mathbb{C})|_s = X_s(\mathbb{C})/H_s(\mathbb{C})$ for $s \in S(\mathbb{C})$. Consider the canonical map $|\pi| : |\mathcal{I}_{\mathcal{X}}(\mathbb{C})| \to |\mathcal{X}(\mathbb{C})|$ of topological spaces over $S(\mathbb{C})$.

Proposition 5.4. *In the above notation, the following holds.*

1. *There is a canonical bijection*

$$|\mathcal{I}_{\mathcal{X}}(\mathbb{C})| = \left\{ \left(x \in X(\mathbb{C}), \gamma \in \mathrm{Stab}_{H_{f(x)}(\mathbb{C})}(x) \right) \right\} /\sim \qquad (6)$$

   where $(x, \gamma) \sim (gx, g\gamma g^{-1})$ for $g \in H_{f(x)}(\mathbb{C})$.

2. *Fix $s \in S(\mathbb{C})$, and consider the induced map $|\pi|_s : |\mathcal{I}_{\mathcal{X}}(\mathbb{C})|_s \to |\mathcal{X}(\mathbb{C})|_s$. Fix $x \in |\mathcal{X}(\mathbb{C})|_s$. There is a canonical bijection*

$$|\pi|_s^{-1}(x) = \left(\coprod_{y \in q_s^{-1}(x)} \Gamma_y \right) /\Gamma \qquad (\Gamma = H_s(\mathbb{C}), \quad \Gamma_y = \mathrm{Stab}_\Gamma(y)). \qquad (7)$$

   Here, $g \in \Gamma$ acts on $\bigsqcup_{y \in q_s^{-1}(x)} \Gamma_y$ as follows: for $y \in q^{-1}(x)$, $\gamma \in \Gamma_y$, we define

$g \cdot (y,\gamma) = (gy, g\gamma g^{-1})$. In particular, for fixed $y' \in q_s^{-1}(x)$, there are bijections

$$|\pi|_s^{-1}(x) = \left(\coprod_{y \in q_s^{-1}(x)} \Gamma_y \right) / \Gamma \cong \Gamma_{y'}/\Gamma_{y'}, \tag{8}$$

of which the second one is in general non-canonical.

Proof. Let $\mathrm{Stab}_S \to X$ be the stabilizer group scheme attached to the action of H on X over S. Then $\mathrm{Stab}_S(\mathbb{C}) = \{(x,\gamma) \in X(\mathbb{C}) \times_{S(\mathbb{C})} H(\mathbb{C}) \mid \gamma x = x\}$. The group scheme H acts on the scheme Stab_S over S by $g \cdot (x,\gamma) = (gx, g\gamma g^{-1})$ for $(g,(x,\gamma)) \in H \times_S \mathrm{Stab}_S$. We have a canonical isomorphism of stacks $\mathcal{I}_{\mathcal{X}} = [\mathrm{Stab}_S/H]$ (see e.g. [Alp25, Exercise 3.2.12]). In particular, (6) follows. This proves item 1. Then (7) follows from item 1. It remains to provide the second bijection in (8). This holds, since for each $y_1, y_2 \in q^{-1}(x)$, there exists $g \in \Gamma$ such that $gy_1 = y_2$ and $g\Gamma_{y_1}g^{-1} = \Gamma_{y_2}$. □

5.2.2 Fibres of the map to the real locus of the coarse moduli space. The next proposition allows one to understand the fibres of the map on real loci $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ induced by the coarse moduli space map $\mathcal{X} \to M$ of a real DM stack $\mathcal{X}$.

Proposition 5.5. *Let $\mathcal{X}$ be a real DM stack, with coarse moduli space $p\colon \mathcal{X} \to M$. Let $f\colon |\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ denote the map induced by p, and let $x \in \mathcal{X}(\mathbb{R})$ with isomorphism class $[x] \in |\mathcal{X}(\mathbb{R})|$ (cf. Section 2).*

1. *There is a canonical bijection $f^{-1}(f([x])) = \mathrm{H}^1(G, \mathrm{Aut}(x_{\mathbb{C}}))$.*
2. *We have $\#\mathrm{H}^1(G, \mathrm{Aut}(x_{\mathbb{C}})) = \#\mathrm{H}^1(G, \mathrm{Aut}(x'_{\mathbb{C}}))$ for each pair of objects $x, x' \in \mathcal{X}(\mathbb{R})$ whose induced objects $x_{\mathbb{C}}, x'_{\mathbb{C}} \in \mathcal{X}(\mathbb{C})$ are isomorphic in $\mathcal{X}(\mathbb{C})$.*

Proof. Since two objects in $\mathcal{X}(\mathbb{C})$ are isomorphic if and only if their images in $M(\mathbb{C})$ are the same, the second item is a consequence of the first item. The first item follows from [Gro60, Section 4]. □

5.3 Topological groupoids and families of finite G-groups. Let $\pi\colon H \to U$ be a locally trivial family of finite topological G-groups. This means that π is a finite topological covering, that there are involutions $\sigma\colon H \to H, \sigma\colon U \to U$ commuting with π, and that there is a continuous group law $m\colon H \times_U H \to H$, an inversion $i\colon H \to H$ and identity $e\colon U \to H$ all compatible with the involutions σ; moreover, we require that for each $x \in U$ there exists an open neighbourhood $x \in V \subset U$ such that $H|_V \cong V \times \Gamma$ as families of topological groups, for a finite group Γ.

Definition 5.6. Let, as above, $\pi\colon H \to U$ be a locally trivial family of finite topological G-groups. We define

$$\begin{aligned} \mathrm{Z}^1(G,H) &\coloneqq \left\{(u,g) \in U \times H \mid u \in U^G, g \in H_u \text{ and } g\sigma(g) = e\right\}, \\ \mathrm{H}^1(G,H) &\coloneqq \mathrm{Z}^1(G,H)/\sim \end{aligned}$$

where $(u,g) \sim (u',g')$ if $u = u'$ and there exists $h \in H_u$ such that $g' = hg\sigma(h)^{-1}$. We equip $\mathrm{Z}^1(G,H)$ with the subspace topology coming from $U \times H$ and we equip $\mathrm{H}^1(G,H)$ with the quotient topology coming from $\mathrm{Z}^1(G,H)$.

Remark 5.7. Note that $\pi\colon H \to U$ defines a topological groupoid $\mathscr{X} = [H \rightrightarrows U]$ enhanced with a natural involution $\sigma\colon \mathscr{X} \to \mathscr{X}$. The maps $s, t\colon H \to U$ are the same and both equal π, and the composition map $c\colon H \times_U H \to H$ equals the group law morphism m. Moreover, by construction, the space $\mathrm{H}^1(G,H)$ is nothing but the coarse space $\left|\mathscr{X}^G\right|$ of the groupoid of fixed points $\mathscr{X}^G$ constructed in Definition 3.4.

Lemma 5.8. *Let $\pi\colon H \to U$ be a locally trivial family of finite topological G-groups. There is a natural surjective map $\mathrm{H}^1(G,H) \to U^G$, and for $V \subset U^G$ open such that $H|_V \cong V \times \Gamma$ as families of topological G-groups over V for some finite G-group Γ, we have $\mathrm{H}^1(G,H)|_V = \mathrm{H}^1(G,H|_V) \cong \mathrm{H}^1(G,\Gamma) \times V$. In particular, the natural map*

$$\mathrm{H}^1(G,H) \longrightarrow U^G$$

is a topological covering, with fibre $\mathrm{H}^1(G,H_u)$ for $u \in U^G$.

Proof. This follows from the fact that the construction of $\mathrm{H}^1(G,H)$ commutes with base change along a map $V \to U$ of topological G-spaces. □

Proposition 5.9. *Let $H \to U$ be a finite étale group scheme over a scheme U of finite type over $\mathbb{R}$. Consider the associated quotient stack $[U/H]$, and also the associated locally trivial family of finite G-groups $H(\mathbb{C}) \to U(\mathbb{C})$. There is a canonical homeomorphism*

$$|[U/H](\mathbb{R})| \xrightarrow{\sim} \mathrm{H}^1(G,H(\mathbb{C}))$$

of spaces over $U(\mathbb{R})$, where the space on the right is defined in Definition 5.6.

Proof. As the group scheme $H \to U$ is finite étale, the morphism $H(\mathbb{C}) \to U(\mathbb{C})$ is a locally trivial family of finite topological G-groups. Hence, in view of Remark 5.7, the proposition is a special case of Theorem 4.8. □

5.4 Gerbes and topological coverings on real loci. In this section we prove Theorem 1.5. Before we can start with the proof, we need three preliminary results.

Lemma 5.10. *Let $f\colon X \to Y$ be a morphism of schemes X, Y which are locally of finite type over $\mathbb{R}$. If f is étale, then $f_{\mathbb{R}}\colon X(\mathbb{R}) \to Y(\mathbb{R})$ is a local homeomorphism.*

Proof. Consider the map of complex analytic spaces $f_{\mathbb{C}}\colon X(\mathbb{C}) \to Y(\mathbb{C})$. This map is a local homeomorphism by [Gro71, Exposé XII, Proposition 3.1 & Remarque 3.3]. Hence the same holds for the restriction to real points and the lemma follows from this. □

Lemma 5.11. *Let $f\colon X \to Y$ be a map of topological spaces, let $\pi\colon Y' \to Y$ be a local homeomorphism with $\mathrm{Im}(\pi) = \mathrm{Im}(f)$. Assume that the base change $f'\colon X' := X \times_Y Y' \to Y'$ is open and a topological covering over its image. Then f is open and a topological covering over its image.*

Proof. This holds, since, for a morphism of topological spaces, the property of being an open map and a topological covering over its image is local on the target. □

For an algebraic stack $\mathcal{X}$, let $\mathcal{X}_{\mathrm{red}} \subset \mathcal{X}$ denote its reduction (cf. [Stacks, Tag 050C]).

Lemma 5.12. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over $\mathbb{R}$. Let $\mathcal{X}_{\mathrm{red}} \subset \mathcal{X}$ be the reduction of $\mathcal{X}$. Let $\mathcal{X} \to M$ and $\mathcal{X}_{\mathrm{red}} \to N$ be the coarse moduli spaces. The natural maps $|\mathcal{X}_{\mathrm{red}}(\mathbb{R})| \to |\mathcal{X}(\mathbb{R})|$ and $N(\mathbb{R}) \to M(\mathbb{R})$ are homeomorphisms.*

Proof. Recall first that $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ is a closed immersion, hence fully faithful (cf. [Stacks, Tag 0504], [Stacks, Tag 04ZZ]). In particular, the map $|\mathcal{X}_{\mathrm{red}}(\mathbb{R})| \to |\mathcal{X}(\mathbb{R})|$ is injective. Let U be a scheme and let $U \to \mathcal{X}$ be a surjective étale morphism such that $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective (see Theorem 4.6). Let $V = \mathcal{X}_{\mathrm{red}} \times_{\mathcal{X}} U$. Then $V \to \mathcal{X}_{\mathrm{red}}$ is étale and essentially surjective on real points. In particular, since $\mathcal{X}_{\mathrm{red}}$ is reduced, V is reduced. The map $V \to U$ is a closed immersion. Thus $V = U_{\mathrm{red}}$, the reduction of U, so that $V(\mathbb{R}) = U(\mathbb{R})$. It follows that $|\mathcal{X}_{\mathrm{red}}(\mathbb{R})| \to |\mathcal{X}(\mathbb{R})|$ is surjective, hence bijective. Moreover, since $V = U_{\mathrm{red}}$, the bijection $|\mathcal{X}_{\mathrm{red}}(\mathbb{R})| \cong |\mathcal{X}(\mathbb{R})|$ is a homeomorphism (cf. Definition 4.7).

Next, we claim that the map $N \to M$ is a universal homeomorphism. The map $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ is a universal homeomorphism [Stacks, Tag 054M] and the map $\mathcal{X} \to M$ is a universal homeomorphism [Stacks, Tag 0DUT]. Thus, the composition $\mathcal{X}_{\mathrm{red}} \to N \to M$, which agrees with the composition $\mathcal{X}_{\mathrm{red}} \to \mathcal{X} \to M$, is also a universal homeomorphism, and hence $N \to M$ is a universal homeomorphism (see the proof of [Stacks, Tag 0H2M]).

It follows that $N \to M$ is universally injective. In particular, the map $N(\mathbb{R}) \to M(\mathbb{R})$ is injective [Stacks, Tag 040X] and the morphism $N \to M$ is radicial [Stacks, Tag 0484]. Since $N \to M$ is radicial and surjective, the map $N(\mathbb{R}) \to M(\mathbb{R})$ is also surjective (see [Stacks, Tag 0481]). We conclude that the map $N(\mathbb{R}) \to M(\mathbb{R})$ is a continuous bijection. Furthermore, the map $\mathcal{X}_{\mathrm{red}} \to N$ is surjective, the map $\mathcal{X}_{\mathrm{red}} \to M$ is proper, and M is separated (see [Con05, Theorem 1.1]). Hence, the map $N \to M$ is proper (see [Stacks, Tag 0CQK]). Consequently, the continuous bijection $N(\mathbb{R}) \cong M(\mathbb{R})$ is closed, and hence a homeomorphism. □

Proof of Theorem 1.5. By Proposition 5.3, we know that $\mathcal{X} \to M$ is a gerbe, and that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is finite étale. The proof proceeds in three steps.

Step 1: *To prove the theorem, we may assume that $\mathcal{X}$ is reduced.* This follows directly from Lemma 5.12.

Step 2: *To prove the theorem, we may assume that $\mathcal{X}$ is reduced and that the coarse moduli space map $\mathcal{X} \to M$ has a section.* To prove this, note that by Step 1, we may assume that $\mathcal{X}$ is reduced. Then $\mathcal{X} \to M$ is a gerbe, see Proposition 5.3. Let $U \to \mathcal{X}$ be a surjective étale morphism where U is a scheme over $\mathbb{R}$, such that $U(\mathbb{R}) \to |\mathcal{X}(\mathbb{R})|$ is surjective. We look at the base change $\mathcal{Y} := \mathcal{X} \times_M U$. The morphism $\mathcal{X} \to M$ is étale, as it is fppf locally on M of the form $[U/H] \to U$ for a finite group scheme $H \to U$ (see [Stacks, Tag 06QH]) and $H \to U$ is étale since $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is étale (see Proposition 5.3). Hence, the composition $U \to \mathcal{X} \to M$ is étale. Therefore, by Lemma 5.10, the map $U(\mathbb{R}) \to M(\mathbb{R})$ is a local homeomorphism, whose image is the image of $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$. Since $\mathcal{X} \times_M \mathcal{X} \to \mathcal{X}$ has a section and $\mathcal{Y} \to U$ is the base change of $\mathcal{X} \times_M \mathcal{X} \to \mathcal{X}$ along $U \to \mathcal{X}$, the morphism $\mathcal{Y} \to U$, which is a gerbe by [Stacks, Tag 06QE], has a section as well. By assumption, the map $|Y(\mathbb{R})| \to U(\mathbb{R})$ is therefore open and a topological covering over its image. As the map $|\mathcal{Y}(\mathbb{R})| \to |\mathcal{X}(\mathbb{R})| \times_{M(\mathbb{R})} U(\mathbb{R})$ is a homeomorphism, we conclude from Lemma 5.11 that the map $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ is open and a topological covering over its image. Step 2 follows.

Step 3: *The map $|\mathcal{X}(\mathbb{R})| \to M(\mathbb{R})$ is a topological covering when $\mathcal{X}$ is reduced and the coarse moduli space map $\mathcal{X} \to M$ has a section.* Indeed, by Proposition 5.3, the map $\mathcal{X} \to M$ is a gerbe. Thus, assuming that $\mathcal{X} \to M$ has a section, we have $\mathcal{X} = [U/H]$ for a scheme U and a group scheme $H \to U$, see [Stacks, Tag 06QG]. Since $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is finite étale (see Proposition 5.3), the map $H \to U$ is finite étale. We have a homeomorphism $|[U/H](\mathbb{R})| \cong \mathrm{H}^1(G, H(\mathbb{C}))$ as spaces over $U(\mathbb{R})$ by Proposition 5.9, and $\mathrm{H}^1(G, H(\mathbb{C})) \to U(\mathbb{R})$ is a topological covering by Lemma 5.8. □

6 Smith–Thom for classifying stacks

Proof of Proposition 1.8. Let $\mathscr{X} := [\Gamma \rightrightarrows \mathrm{pt}]$. By Example 3.6, we have $\left|\mathscr{X}^G\right| \simeq \mathrm{H}^1(G,\Gamma)$, where $\mathrm{H}^1(G,\Gamma)$ is the finite discrete space $\mathrm{Z}^1(G,\Gamma)/\sim$ with $\mathrm{Z}^1(G,\Gamma) \subseteq \Gamma$ the set of $\gamma \in \Gamma$ such that $\gamma\sigma(\gamma) = e$ and $\sim$ the equivalence relation $\gamma \sim \beta\gamma\sigma(\beta)^{-1}$ for $\beta \in \Gamma$. Therefore, we have $\dim \mathrm{H}^*(\left|\mathscr{X}^G\right|, \mathbb{Z}/2) = \#\mathrm{H}^1(G,\Gamma)$. Moreover, by Proposition 5.4, we have $|\mathcal{I}_{\mathscr{X}}| \simeq \Gamma/\Gamma$ so that $\dim \mathrm{H}^*(|\mathcal{I}_{\mathscr{X}}|, \mathbb{Z}/2) = \#(\Gamma/\Gamma)$, where Γ/Γ is set of conjugacy classes of Γ. So Proposition 1.8 follows from the following group theoretic lemma, whose proof has been suggested to us by Will Sawin. ☐

Lemma 6.1. *Let Γ be a finite group with an action of G. Then $\#\mathrm{H}^1(G,\Gamma) \leq \#(\Gamma/\Gamma)$..*

Proof. Let $\sigma\colon \Gamma \to \Gamma$ be the involution corresponding to the G-action. Let σ-conj be the equivalence relation on Γ induced by the action of Γ on its self by σ-conjugacy (i.e. h acts by $h(g) = hg\sigma(h^{-1})$. For every $h \in \Gamma$, we let $\mathrm{Stab}_\sigma(h)$ (resp. $[h]_\sigma$) be the stabilizer (resp. the orbit) of h for the σ-conjugacy action and $\mathrm{Stab}(h)$ (resp. $[h]$), the stabilizer (resp. the orbit) for the conjugacy action. For instance, $\mathrm{Stab}(h) = \left\{g \in \Gamma \mid g^{-1}hg = h\right\}$.

We claim the following chain of inequalities and equalities:

$$\#\mathrm{H}^1(G,\Gamma) \leq \#(\Gamma/\sigma\text{-conj}) = \#(\Gamma/\Gamma)^G \leq \#(\Gamma/\Gamma).$$

Since the first and the last inequalities follow from the inclusions $\mathrm{H}^1(G,\Gamma) \subseteq (\Gamma/\sigma\text{-conj})$ and $(\Gamma/\Gamma)^G \subseteq \Gamma/\Gamma$, we just need to prove the middle equality.

By the orbit-stabilizer theorem, we have $\sum_{g\in[g]_\sigma} \#\mathrm{Stab}_\sigma(g) = \#\Gamma$. Therefore,

$$\sum_{g\in\Gamma} \#\mathrm{Stab}_\sigma(g) = \sum_{[g]_\sigma\in\Gamma/\sigma\text{-conj}} \sum_{g\in[g]} \#\mathrm{Stab}_\sigma(g) = \#\Gamma \cdot \#(\Gamma/\sigma\text{-conj}).$$

Moreover,

$$\begin{aligned}\sum_{g\in\Gamma} \#\mathrm{Stab}_\sigma(g) &= \sum_{g\in\Gamma} \#\left\{h \in \Gamma \mid hg\sigma(h)^{-1} = g\right\} = \sum_{h\in\Gamma} \#\left\{g \in \Gamma \mid \sigma(h) = g^{-1}hg\right\} \\ &= \sum_{[h]\in(\Gamma/\Gamma)^G} \sum_{h\in[h]} \#\left\{g \in \Gamma \mid \sigma(h) = g^{-1}hg\right\} = \sum_{[h]\in(\Gamma/\Gamma)^G} \sum_{h\in[h]} \#\mathrm{Stab}(h) \\ &= \sum_{[h]\in(\Gamma/\Gamma)^G} \#\Gamma = \#(\Gamma/\Gamma)^G \cdot \#\Gamma,\end{aligned}$$

where the penultimate equality follows again from the orbit-stabilizer theorem (now applied to the action of Γ on Γ by conjugation). This concludes the proof. ☐

7 Smith–Thom inequality for groupoid cohomology

In the previous sections, we examined the topological spaces $\left|\mathscr{X}^G\right|$ and $|\mathscr{X}|$ associated to a topological G-groupoid $\mathscr{X} = [R \rightrightarrows U]$, and proposed Conjecture 1.6.

In a complementary direction, it is natural to compare the *groupoid cohomology* of $\mathscr{X}^G$ with the groupoid cohomology of $\mathscr{X}$, as another route to generalizing the Smith–Thom inequality. More precisely, let us suppose that $\mathscr{X}$ is an étale groupoid—that is, the source and target maps $s, t \colon R \to U$ are local homeomorphisms—equipped with an involution $\sigma \colon \mathscr{X} \to \mathscr{X}$. Then, as shown in Lemma 3.9, the fixed-point groupoid $\mathscr{X}^G$ is also étale. In this context, one may consider the groupoid cohomology $\mathrm{H}^*_{\mathrm{grp}}$ of both $\mathscr{X}$ and $\mathscr{X}^G$ (cf. [CM00, Section 2.1]).

Remark that the cohomology group $\mathrm{H}^*_{\mathrm{grp}}(\mathscr{X}, \mathbb{Z}/2)$ is not finite dimensional in general, hence it does not make sense to compare the dimension of $\mathrm{H}^*_{\mathrm{grp}}(\mathscr{X}^G, \mathbb{Z}/2)$ with the dimension of $\mathrm{H}^*_{\mathrm{grp}}(\mathscr{X}, \mathbb{Z}/2)$. Instead, one can ask the following.

Question 7.1. *Let $(\mathscr{X} = [R \rightrightarrows U], \sigma \colon \mathscr{X} \to \mathscr{X})$ be an étale G-groupoid.*

1. *Is there a natural upper bound on the ratio*

$$\frac{\dim \mathrm{H}^{\leq i}_{\mathrm{grp}}(\mathscr{X}^G, \mathbb{Z}/2)}{\dim \mathrm{H}^{\leq i}_{\mathrm{grp}}(\mathscr{X}, \mathbb{Z}/2)}$$

   as $i \to \infty$?

2. *If such a bound exists, can it be made independent of $\mathscr{X}$ (and, in particular, of the involution σ)?*

If $\mathscr{X}$ is a topological orbifold arising as the quotient of a topological space X by the action of a finite group Γ, then by [MP99, Section 1.3], we have an isomorphism

$$\mathrm{H}^i_{\mathrm{grp}}(\mathscr{X}, \mathbb{Z}/2) \simeq \mathrm{H}^i_{\Gamma}(X, \mathbb{Z}/2),$$

where the right-hand side denotes the Γ-equivariant cohomology of X. Moreover, by Example 3.6, if Γ is a finite group with an involution $\sigma \colon \Gamma \to \Gamma$, and $\mathscr{X} = B\Gamma = [\Gamma \rightrightarrows \mathrm{pt}]$ is the classifying groupoid of Γ, then there is a natural isomorphism of topological groupoids

$$\mathscr{X}^G \cong \coprod_{[\gamma] \in \mathrm{H}^1(G, \Gamma)} B(\Gamma^{\sigma_\gamma})$$

where $B(\Gamma^{\sigma_\gamma}) = [\Gamma^{\sigma_\gamma} \rightrightarrows \mathrm{pt}]$ and for an element $\gamma \in \Gamma$ with $\gamma\sigma(\gamma) = e$, the subgroup $\Gamma^{\sigma_\gamma} \subseteq \Gamma$ consists of those $g \in \Gamma$ such that $\gamma\sigma(g) = g\gamma$.

Proposition 7.2. *Question 7.1.1 has a positive answer when* $\mathscr{X} = B\Gamma$ *for a finite* G*-group* Γ*.*

Proof. By [Qui71, Corollary 2.2], the cohomology ring $\mathrm{H}^*(\Gamma, \mathbb{Z}/2)$ is a finitely generated graded $\mathbb{F}_2$-algebra. Consider the associated Poincaré series defined as

$$P_\Gamma(t) \coloneqq \sum_{i=0}^{\infty} \dim_{\mathbb{F}_2} \mathrm{H}^i(\Gamma, \mathbb{Z}/2) \cdot t^i \in \mathbb{Z}[[t]].$$

By a result of Venkov (see [Qui71, Proposition 2.5]), one has that $P_H(t)$ is a rational function for any finite group H. For each $\gamma \in \Gamma$ such that $\gamma\sigma(\gamma) = e$, we obtain a rational function

$$P_{\Gamma,\gamma}(t) \coloneqq \frac{P_{\Gamma^{\sigma_\gamma}}(t)}{P_\Gamma(t)} \in \mathbb{Q}(t). \tag{9}$$

Since $\Gamma^{\sigma_\gamma} \subseteq \Gamma$ is a subgroup, it follows from (see [Qui71, Proposition 2.5, Theorem 7.7]), that the rational function $P_{\Gamma,\gamma}(t)$ has no pole at $t = 1$, and thus the evaluation $P_{\Gamma,\gamma}(1) \in \mathbb{Q}$ is well-defined. Hence, the proposition follows from Lemma 7.3 below. $\square$

Lemma 7.3. *Let* $f(t) = \sum_{i\geq 0} a_i t^i$ *and* $g(t) = \sum_{i\geq 0} g(t)$ *be power series with* $a_i, b_j \in \mathbb{Q}_{\geq 0}$ *and* $a_0 \neq 0$, $b_0 \neq 0$*. For* $N \geq 0$*, define* $C_N = (\sum_{i=0}^N a_i)/(\sum_{i=0}^N b_i) \in \mathbb{Q}_{>0}$*. Assume that* $f(t)$ *and* $g(t)$ *are rational functions. Then the quotient* $h(t) = f(t)/g(t)$ *satisfies* $h(1) = \lim_{N\to\infty} C_N$ *provided that* $h(t)$ *has no pole at* $t = 1$*.*

Proof. Since $f(t), g(t) \in \mathbb{Q}(t)$, we can write $f(t) = \frac{P(t)}{(1-t)^r Q(t)}$ and $g(t) = \frac{R(t)}{(1-t)^s S(t)}$ where $P(t), Q(t), R(t), S(t) \in \mathbb{Q}[t]$, and $Q(1), S(1) \neq 0$. The integers $s \geq r \geq 0$ are the pole orders at $t = 1$. From [FS09, Theorem VI.1, p. 381], we know that $a_n \sim \frac{P(1)}{Q(1)} \cdot \frac{n^{r-1}}{(r-1)!}$ hence, by Faulhaber's formula, $A_N \coloneqq \sum_{n=0}^N a_n \sim \frac{P(1)}{Q(1)} \cdot \frac{N^r}{r!}$. Likewise, $B_N \coloneqq \sum_{n=0}^N b_n \sim \frac{R(1)}{S(1)} \cdot \frac{N^s}{s!}$ so that

$$\frac{A_N}{B_N} \sim \frac{P(1)S(1)}{Q(1)R(1)} \cdot \frac{(s-1)!}{(r-1)!} \cdot N^{r-s} \sim \begin{cases} 0 & \text{if } s > r \\ \frac{P(1)S(1)}{Q(1)R(1)} & \text{if } r = s. \end{cases}$$

In both cases, $A_N/B_N \sim h(1)$, proving the lemma. $\square$

In fact, the proof of Proposition 7.2 shows something more precise. Let Γ be a finite group and $\sigma\colon \Gamma \to \Gamma$ be an involution. For $\gamma \in \Gamma$ with $\gamma\sigma(\gamma) = e$, define $P_{\Gamma,\gamma}(t) \in \mathbb{Q}(t)$ as in (9). Heuristically, the value $P_{\Gamma,\gamma}(1) \in \mathbb{Q}$ reflects the ratio between the total mod 2 Betti numbers of $B(\Gamma^{\sigma_\gamma})$ and $B\Gamma$.

Corollary 7.4. *Let Γ be a finite G-group and let $\mathscr{X} = B\Gamma$. Then, we have:*

$$\varinjlim_{i\to\infty}\left(\frac{\dim \mathrm{H}^{\leq i}_{\mathrm{grp}}(\mathscr{X}^G,\mathbb{Z}/2)}{\dim \mathrm{H}^{\leq i}_{\mathrm{grp}}(\mathscr{X},\mathbb{Z}/2)}\right) = \sum_{[\gamma]\in \mathrm{H}^1(G,\Gamma)} \varinjlim_{i\to\infty}\left(\frac{\dim \mathrm{H}^{\leq i}(\Gamma^{\sigma_\gamma},\mathbb{Z}/2)}{\dim \mathrm{H}^{\leq i}(\Gamma,\mathbb{Z}/2)}\right) = \sum_{[\gamma]\in \mathrm{H}^1(G,\Gamma)} P_{\Gamma,\gamma}(1).$$

This motivates the following weaker version of Question 7.1.2.

Question 7.5. *Does there exist a constant $C > 0$ such that for every finite group Γ and involution $\sigma\colon \Gamma \to \Gamma$, we have*

$$\sum_{[\gamma]\in \mathrm{H}^1(G,\Gamma)} P_{\Gamma,\gamma}(1) \leq C\ ?$$

A positive answer to Question 7.5 would provide a stepping stone toward a general positive answer to Question 7.1.2. However, even this zero-dimensional case is already quite subtle. For instance, the following example shows that one cannot take $C = 1$.

Example 7.6. Let $\mathfrak{S}_4$ denote the symmetric group on four letters, equipped with the trivial G-action. The elements $\gamma_1 := e, \gamma_2 := (12)$ and $\gamma_3 := (12)(34)$ form a complete set of representatives for the equivalence classes in $\mathrm{H}^1(G,\mathfrak{S}_4)$. In particular, $\#\mathrm{H}^1(G,\mathfrak{S}_4) = 3$. The corresponding fixed-point subgroups are:

$$\mathfrak{S}_4^{\gamma_1} = \mathfrak{S}_4, \qquad \mathfrak{S}_4^{\gamma_2} = \{e,(12),(34),(12)(34)\} \simeq \mathbb{Z}/2\times\mathbb{Z}/2,$$

$$\mathfrak{S}_4^{\gamma_3} = \{e,(12),(34),(12)(34),(13)(24),(14)(23),(1423),(1324)\} \simeq D_8,$$

where D_8 denotes the dihedral group of order 8.

From [Nak62, Theorem 4.1] and [Han93, Theorem 5.5], one computes the corresponding Poincaré series:

$$P_{\mathbb{Z}/2\times\mathbb{Z}/2}(t) = \frac{1}{(1-t)^2}, \quad P_{D_8}(t) = \frac{1}{(1-t)^2}, \quad P_{\mathfrak{S}_4}(t) = \frac{1+t^2}{(1-t)^2(1+t+t^2)}.$$

Evaluating the associated ratios at $t = 1$, we obtain $P_{\mathfrak{S}_4,\gamma_1}(1) = 1$ and $P_{\mathfrak{S}_4,\gamma_2}(1) = P_{\mathfrak{S}_4,\gamma_3}(1) = 3/2$. Consequently, we have $\sum_{[\gamma]\in \mathrm{H}^1(G,\mathfrak{S}_4)} P_{\mathfrak{S}_4,\gamma}(1) = 1 + \frac{3}{2} + \frac{3}{2} = 4$.

References

[AGF25] Emiliano Ambrosi and Olivier de Gaay Fortman. *On the topology of real algebraic stacks*. 2025.

[Alp25] Jarod Alper. *Stacks and Moduli*. `https://sites.math.washington.edu/~jarod/moduli.pdf`. 2025.

[AV02] Dan Abramovich and Angelo Vistoli. "Compactifying the space of stable maps". In: *J. Amer. Math. Soc.* 15.1 (2002), pp. 27–75.

[Bor60] Armand Borel. *Seminar on transformation groups*. Annals of Mathematics Studies, No. 46. With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Princeton University Press, Princeton, NJ, 1960, pp. vii+245.

[BS22] Erwan Brugallé and Florent Schaffhauser. "Maximality of moduli spaces of vector bundles on curves". In: *Épijournal Géom. Algébrique* 6 (2022), Art. 24, 15.

[CM00] Marius Crainic and Ieke Moerdijk. "A homology theory for étale groupoids". In: *J. Reine Angew. Math.* 521 (2000), pp. 25–46.

[Con05] Brian Conrad. *The Keel–Mori theorem via stacks*. `https://math.stanford.edu/~conrad/papers/coarsespace.pdf`. 2005.

[DIK00] Alex Degtyarev, Ilia Itenberg, and Viatcheslav Kharlamov. *Real Enriques surfaces*. Vol. 1746. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000, pp. xvi+259.

[Eti+10] Pavel Etingof et al. "The cohomology ring of the real locus of the moduli space of stable curves of genus 0 with marked points". In: *Ann. of Math. (2)* 171.2 (2010), pp. 731–777.

[Flo52] Edwin Earl Floyd. "On periodic maps and the Euler characteristics of associated spaces". In: *Trans. Amer. Math. Soc.* 72 (1952), pp. 138–147.

[Fra18] Matthias Franz. "Symmetric products of equivariantly formal spaces". In: *Canad. Math. Bull.* 61.2 (2018), pp. 272–281.

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009, pp. xiv+810.

[Fu23] Lie Fu. *Maximal real varieties from moduli constructions*. 2023. arXiv: `2303.03368 [math.AG]`.

[GF22a] Olivier de Gaay Fortman. "Moduli spaces and algebraic cycles in real algebraic geometry". PhD thesis. École normale supérieure de Paris, 2022.

[GF22b] Olivier de Gaay Fortman. "Real moduli spaces and density of non-simple real abelian varieties". In: *Q. J. Math.* 73.3 (2022), pp. 969–989.

[GF24] Olivier de Gaay Fortman. "On the integral Hodge conjecture for real abelian threefolds". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2024.807 (2024), pp. 221–255. URL: `https://doi.org/10.1515/crelle-2023-0082`.

[GH81] Benedict Gross and Joe Harris. "Real algebraic curves". In: *Ann. Sci. École Norm. Sup. (4)* 14.2 (1981), pp. 157–182.

[Gro60] Alexander Grothendieck. "Technique de descente et théorèmes d'existence en géométrie algébrique. I. Généralités. Descente par morphismes fidèlement plats". In: *Séminaire Bourbaki, Vol. 5*. Soc. Math. France, Paris, 1960, Exp. No. 190, 299–327.

[Gro71] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*. Vol. 224. Lecture notes in mathematics. Springer-Verlag, 1971.

[Han93] David Handel. "On products in the cohomology of the dihedral groups". In: *Tohoku Math. J. (2)* 45.1 (1993), pp. 13–42.

[KM97] Seán Keel and Shigefumi Mori. "Quotients by groupoids". In: *Ann. of Math. (2)* 145.1 (1997), pp. 193–213.

[Knu71] Donald Knutson. *Algebraic spaces*. Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin-New York, 1971, pp. vi+261.

[KR24] Viatcheslav Kharlamov and Rareş Răsdeaconu. *Unexpected loss of maximality: the case of Hilbert square of real surfaces*. 2024. arXiv: `2303.02796 [math.AG]`.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*. Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer-Verlag, Berlin, 2000.

[Man17] Frédéric Mangolte. *Variétés Algébriques Réelles.* Vol. 24. Cours Spécialisés. Société Mathématique de France, 2017, pp. vii+484.

[MP99] Ieke Moerdijk and Dorette Pronk. “Simplicial cohomology of orbifolds”. In: *Indag. Math. (N.S.)* 10.2 (1999), pp. 269–293.

[Nak62] Minoru Nakaoka. “Note on cohomology algebras of symmetric groups”. In: *J. Math. Osaka City Univ.* 13 (1962), pp. 45–55. ISSN: 0449-2773.

[Noo12] Behrang Noohi. “Homotopy types of topological stacks”. In: *Adv. Math.* 230.4-6 (2012), pp. 2014–2047.

[Qui71] Daniel Quillen. “The spectrum of an equivariant cohomology ring. I, II”. In: *Ann. of Math. (2)* 94 (1971), 549–572; ibid. (2) 94 (1971), 573–602.

[SS89] Mika Seppälä and Robert Silhol. “Moduli spaces for real algebraic curves and real abelian varieties”. In: *Math. Z.* 201.2 (1989), pp. 151–165.

[Stacks] The Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`. 2018.

[Tho65] René Thom. “Sur l’homologie des variétés algébriques réelles”. In: *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse).* Princeton Univ. Press, Princeton, N.J., 1965, pp. 255–265.

[Wil25] Sawin Will. *Answer to a question on Mathoverflow.* 2025. URL: `https://mathoverflow.net/questions/485688/a-finite-group-gamma-with-an-involution-such-that-vert-h1-mathbb-z-2-math`.

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