ℓ-adic, p-adic and geometric invariants in families of varieties

Ph.D. Defense

Emiliano Ambrosi

École Polytechnique

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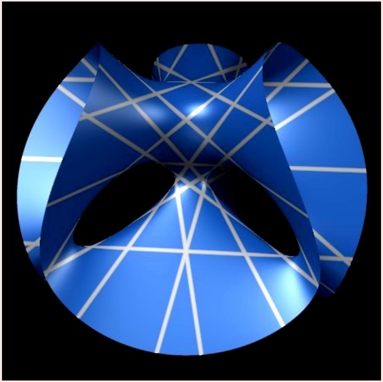
Cubic surface



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Cycle class map

$c_Y : \mathrm{CH}^i(Y) \otimes \mathbb{Q} \to \mathrm{H}^{2i}(Y)$ relates geometry to cohomology.

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$$\begin{array}{ccccc} Y_{\overline{x}} & \longrightarrow & Y_x & \longrightarrow & Y \\ \downarrow & \square & \downarrow & \square & \downarrow f \\ \overline{k(x)} & \longrightarrow & k(x) & \xrightarrow{x} & X \end{array}$$

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- Extra structures (Hodge filtrations, Galois actions, Frobenius actions) can be very different!
- Also $\mathrm{NS}(Y_x) \otimes \mathbb{Q}$, $\mathrm{Pic}(Y_x) \otimes \mathbb{Q}$, $\mathrm{CH}^i(Y_x) \otimes \mathbb{Q}$ vary.

NS-generic points

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Y_{\overline{x}} & \longrightarrow & Y & \longleftarrow & Y_{\overline{\eta}} \\
\downarrow & \square & \downarrow f & \square & \downarrow \\
\overline{k(x)} & \xrightarrow{\overline{x}} & X & \xleftarrow{\overline{\eta}} & \overline{k(\eta)}
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- Injective specialization morphism:

$$sp_{\eta,x} : \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \hookrightarrow \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q};$$

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Definition

x NS-generic (resp. arithmetically NS-generic) if $sp_{\eta,x}$ (resp. $sp^{ar}_{\eta,x}$) isomorphism.

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The answers depend on the arithmetic of k.

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- x is NS-generic $\Leftrightarrow$ $Y_{\overline{x}}$ has not complex multiplication.
- $k = \mathbb{F}_q$ finite field $\Rightarrow$ $Y_{\overline{x}}$ has always complex multiplication.

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Example 2

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Example 2.0

- Veronese's embedding of degree 2

$$\mathbb{P}^3_k \to \mathbb{P}^9_k$$
$$[x : y : z : w] \mapsto [x^2 : y^2 : z^2 : w^2 : xy : xz : xw : yz : yw : zw];$$

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- Hyperplane section $Y_x \leftrightarrow$ Quadric $Q_x \subseteq \mathbb{P}^3$;
- $k = \overline{k} \Rightarrow Q_x \simeq \mathbb{P}^1 \times \mathbb{P}^1$,

$$\mathrm{NS}(Q_x) \otimes \mathbb{Q} \simeq \mathbb{Q} \times \mathbb{Q}, \quad \text{while} \quad \mathrm{NS}(\mathbb{P}^3) \otimes \mathbb{Q} \simeq \mathbb{Q}.$$

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Remark (p=0)

If $p = 0$:

1. is due to André;
2. is due to Cadoret-Tamagawa.

Tate conjecture

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Cycles class map

$$c_Y : \mathrm{NS}(Y) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1))$$

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Conjecture (Tate)

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$$\pi_1(k(\eta)) \xrightarrow{\rho_{\ell,\eta}} GL(\mathrm{H}^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$$

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Inclusion of ℓ-adic Lie groups

$$\rho_\ell(\pi_1(k(x))) =: \Pi_{\ell,x} \subseteq \Pi_\ell := \rho_\ell(\pi_1(X))$$

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$p > 0$, k finitely generated, X curve $\Rightarrow$ all but finitely many $x \in X(k)$ Galois generic.

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If $p = 0$ Theorem 2 is due to Cadoret-Tamagawa.

Proof of Theorem 2

Anabelian dictionary

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- $U \subseteq \Pi_\ell$ open subgroup $\leftrightarrow$ connected étale cover $X_U \to X$
- $\Pi_{\ell,x} \subseteq U \Leftrightarrow k(x) \xrightarrow{x} X$ lifts to a $k(x)$-rational point of X_U.

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Construction (Cadoret-Tamagawa)

$\exists$ projective system $h_n : \mathcal{X}_n \to X$ of étale covers such that Theorem 2 holds

$$\Leftrightarrow \qquad \mathrm{Im}(\varprojlim_n(\mathcal{X}_n(k)) \to X(k)) \text{ finite}$$

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Mordell conjecture (Samuel-Voloch)

Genus $g_{\mathcal{X}_n}$ of $\mathcal{X}_n \gg 0 \Rightarrow \mathcal{X}_n(k)$ finite.

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Tate conjecture predicts:

(Strictly) Galois generic points are (arithmetically) NS-generic.

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(2) Comparison étale-singular sites, to link Hodge theory to ρ_ℓ.

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Replacements

(1) replaced with the crystalline variational Tate conjecture.

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\mathrm{H}^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) & \longrightarrow & \mathrm{H}^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)) \\
\downarrow \text{(surjective)} & \nearrow \text{(injective)} & \\
\mathrm{H}^0(X_{\overline{k}}, R^2 f_* \mathbb{Q}_\ell(1)) & &
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\mathrm{Pic}(Y) \otimes \mathbb{Q} & \longrightarrow & \mathrm{NS}(Y_x) \otimes \mathbb{Q} \\
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\mathrm{H}^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) & \longrightarrow & \mathrm{H}^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)) \\
\twoheaddownarrow & \nearrow & \\
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\end{array}$$

Galois generic assumption $\Rightarrow$

$\mathrm{H}^0(X_{\overline{k}}, R^2 f_* \mathbb{Q}_\ell(1))^{\pi_1(k)} \simeq \mathrm{H}^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))^{\pi_1(k)}$

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Crystalline Variational Tate conjecture (Morrow):

$$Im(\mathrm{Pic}(Y)\otimes\mathbb{Q} \to \mathrm{NS}(Y_x)\otimes\mathbb{Q}) = \mathrm{H}^0(X, R^2 f_{crys,*}\mathcal{O}_{Y/K}(1))^F \cap \mathrm{NS}(Y_x)\otimes\mathbb{Q}$$

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x strictly Galois generic $\Rightarrow \mathrm{H}^0(X, R^2 f_{crys,*}\mathcal{O}_{Y/K}(1))^F \simeq \mathrm{H}^2_{crys}(Y_x)(1)^F$

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Category **F-Isoc**(X) of F-isocrystals has a pathological behaviour:

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Replace $K\{T\}$ with

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  - global monodromy theorem (Crew, Kedlaya).

From crystals to overconvergent F-isocrystals

Fact

- *There is a functor Forg* : $\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(X)$ *(Berthelot-Ogus);*

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$f : Y \to X$ smooth and proper, $R^{i} f_{crys,*} O_{Y/K}$ is the image of a $R^{i} f_{*} O^{\dagger}_{Y/K} \in \mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(X)$.

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Consequence:

Enough to compare $R^2 f_* O^{\dagger}_{Y/K}(1)$ and $R^2 f_* \mathbb{Q}_\ell(1)$.

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Proposition

$G(x^*\mathcal{F}) = G(\mathcal{F})$ if and only if $G(x^*\mathcal{E}) = G(\mathcal{E})$

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Perfect p-torsion of abelian varieties (joint with D'Addezio)

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- Transfer this information comparing the (maximal tori in the) monodromy groups of $\mathcal{E}^\dagger$ and $\mathcal{E}$.

THANK YOU FOR THE ATTENTION!