An explicit bound on ℓ-primary torsion of one dimensional families of abelian varieties in positive characteristic

Emiliano Ambrosi

Explicit and computational approaches to Galois representations, Luxembourg
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Problem:

Study the variation of $Y_x(k(x))[\ell^\infty]$ with $x \in |X|$.

Theorem (E.A.)

Assume that $Y \to X$ admits an "$\mathbb{F}_{p^n}$-model". Then there exists an explicit constant $C := C(Y \to X, \ell)$ such that for $n \geq C$ there are only finitely many $x \in X(k)$ such that $Y_x[\ell^\infty](k)$ contains a point of order $\geq \ell^n$.

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Remark

We will not make precise the hypothesis of having "$\mathbb{F}_{p^n}$-model". It is always true up to a finite base change and it is used to reduce the computations to finite fields.

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- $\rho_{\ell,x} : \pi_1(x) \to \pi(X) \to GL(T_{\ell^\infty}(A_{\overline{\eta}}));$
- $\rho_{\ell,x}$ identifies with the natural representation

$$\pi_1(x) \to GL(T_{\ell^\infty}(A_{\overline{x}})).$$

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Lemma

A_x has a $k(x)$-point of exact order ℓ^n if and only if $x \in |X|$ lifts to a $k(x)$-rational point of $X_1(\ell^n)$

$$\begin{array}{ccc} & & X_1(\ell^n) \\ & \nearrow & \downarrow \\ Spec(k(x)) & \longrightarrow & X \end{array}$$

Abstract modular curves

Proposition

There exists an explicit constant $C := C(Y \to X, \ell)$ such that for $n \geq C$ the set $X_1(\ell^n)(k)$ is finite.

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Remark

The constant g *is easy to compute.*

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Definition

C smooth proper connected curve, the k-gonality of C, $\gamma_{C,k}$ is

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Lemma

If C has a rational point then $g_C \geq \gamma_{C,k} - 1$.

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Lemma (Cadoret-Tamagawa)

Fix $d \in \mathbb{Z}$. If $D_C \subseteq \mathbb{Z}_{\geq d} \cup \bigcup_{n \geq 0} \ell^n$ then $\gamma_C \geq \frac{d}{2}$

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- So $[k(c) : k(x)]2 \in D_C$ but $[k(c) : k(x)]2 < d$.
- This is in contradiction with $D_C \subseteq \mathbb{Z}_{\geq d} \cup \bigcup_{n \geq 0} \ell^n$.

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- Thanks to the lemma, it is enough to control the following quantity:
- $[k(x_n) : k]$ for $x_n \in X_{m_n}$.
- Replacing X with a finite étale cover, we may assume that the image Π_{ℓ^∞} of ρ_ℓ is pro-ℓ and that $x \in |X^{cmp}| - |X|$ are k-rational.

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- and so

$$[k(x_n):k] \geq 2\frac{\ln(\ell^{n/2g}-1)}{\ln(|k|)}$$