

REDUCTION MODULO p OF

NOETHER'S PROBLEM (WORK IN PROGRESS WITH DOMENICO VALLO)

① NOETHER'S PROBLEM AND SPECIALISATION

k FIELD, $\text{CHAR}(k) = p \geq 0$, l PRIME NUMBER.
" \bar{k}

DEF • X, Y k -VARIETIES (IRREDUCIBLES)

$X \stackrel{SB}{\sim} Y$ (X IS STABLY BIRATIONAL TO Y)

IF $\exists m, n \in \mathbb{N}_{\geq 0}$ SUCH THAT.

$X \times \mathbb{P}^m \stackrel{B}{\sim} Y \times \mathbb{P}^n$ (BIRATIONAL TO)

- $SB(k) = \{ \text{VARIETIES } / k \} / \stackrel{SB}{\sim}$
- $X/k \rightsquigarrow [X]_k \in SB(k)$
- X IS STABLY RATIONAL IF $[X]_k = [0]_k$
- FINITE (l -GROUP) $G \subseteq GL(V)$

V FIN DIMENSIONAL k -VECTOR SPACE

$\left[\frac{V}{G} \right]_k$ DOES NOT DEPEND ON THE CHOICE OF GRV (NO-NAME LEMMA)

NOETHER'S PROBLEM (1917):

$$\text{IS } \left[\frac{V}{G} \right]_k = [0]_k ?$$

ANSWER:

- NO IN GENERAL
- EXAMPLES WITH $l \neq p$
- SALTMAN '83 $|G| = l^2$
 - BOUMOLOU '87 $|G| = l^6$
 - PETRE '08 $|G| = l^{12}$
- DIFFERENT NATURE

• YES SOMETIMES:

- G ABELIAN, $p \nmid |G|$ SWAN '83 FISHER '75
- $|G| = p^m$ ($p = \text{CHAR}(k)$)
KUNIKOSHI '59

UPSHOT: GEOMETRY OF V/\mathcal{G}
DEPENDS ON $\text{char}(k)$.

IN BIRATIONAL GEOMETRY ARE USED
SPECIALISATION TECHNIQUES, PUTTING
VARIETIES IN FAMILIES AND STUDYING
CHANGES OF GEOMETRY (KOLLAR, TATARU,
VAININ, HASSOT,
CALLIOT-THÉLÈNE,
PIRUTKA, SCHABINGERER
.....)

Q: CAN WE INTERPOLATE V/\mathcal{G} BETWEEN
CHARACTERISTIC ZERO AND p ?

Q(a): \exists COMPLETE VAL. RING (R, \mathfrak{m}) OF
MIXED CHARACTERISTIC (q, p) ($p > q$) AND
A SMOOTH PROPER R -SCHEME $X \rightarrow \text{Spec}(R)$.
SUCH THAT $X_k = [a]_k$ AND ?
 $X_k = [a]_k$

$$k = \bar{k} = R/m \quad k = \bar{k} = \text{FRAC}(R)$$

THM 1(AV) THE ANSWER TO Q(A) IS NO,

THE GROUPS CONSTRUCTED FROM SAUTMAN AND BONGIORNO VARI

RMK BIRATIONAL TREES CAN VARY IN FAMILIES (EQUIVARIANT)
 (STABIN) (UJISIN, SHIMMER, NICAISE, SCHLAFER) MIXED FIBRATIONS + HASSET TAVANNO

(2) (STABLE) - BIRATIONAL INVARIANTS

X/k SMOOTH AND PROPER.

FRAC COMB. TO COEFFICIENTS OF SERIES CUBIC FORMS

EX 1 $h^i(A) := D(1) \left(H^0(X, \mathcal{O}^i) \right)$

IS A STABLE BIRATIONAL INVARIANT (HARDY) (EASY)

$h^i(A) \Rightarrow \left. \begin{array}{l} = h^i(A) \text{ IN CHAR } \neq 0. \\ \text{CHARACTERIZATION -} \\ \text{RÜLLING IN CHAR } 0 \end{array} \right\}$

RMK $p=2$ $A \rightarrow R$ ABELIAN SURFACE

A_p SUPER SINGULAR

$kum(A_k) :=$ MINIMAL RESOLUTION OF $A_k / [-1]$ IS A $k=3$ $(H^0(X, \mathcal{O}^2) \neq 0)$

RATIONAL VANITY

• $KUM(A_k) :=$ II (SHIMURA, '94, KATZUNA, '99) ($H^0(X, \mathcal{O}_X^2) = 0$)

$\nexists X \rightarrow \mathbb{R}$ SMOOTH PROPER SUCH THAT

$$[X_k] = [KUM(A_k)] \quad \text{AND} \quad [X_k] = [KUM(A_k)]$$

BY SEMICONTINUITY $(\dim H^0(X_k, \mathcal{O}_X^2) \geq \dim H^0(X_k, \mathcal{O}_X^2))$

LAZDA-SKAROBAGANOV: SUCH A MODEL $X \rightarrow \mathbb{R}$ EXISTS IF A_k IS NOT SUPERSIMULAR.

EX 2 $AM_2(X)$ ARTIN-MUMFORD INVARIANT

$$L \neq P \quad \dim_{\mathbb{F}} H^3_{\text{BT}}(X|L) \neq \dim_{\mathbb{F}} H^3_{\text{BT}}(X|P) \quad \left(\frac{AR(X)}{BR(X|DIB)}, [L] \right)$$

(5) A RATIONAL INVARIANT.

SALMAN-BONNOMERON. EXAMPLES:

$$AM_2(\mathbb{A}^1/\mathbb{G}) \neq 0 \quad (\text{NOT IN PERRE EXAMPLES})$$

THM 1' (AV) $X \rightarrow \mathbb{R}$ SMOOTH AND PROPER

IF x_k IS RATIONAL $H^3(x_k, \mathbb{Z}(p))(\mathbb{P}) = 0$.

(THM 1' \Rightarrow THM 1)

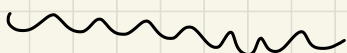
③ INTEGRAL p -ADIC HODGE THEORY

IF $l \neq p$ $H^i(x_k, \mathbb{Z}(l)) \cong H^i(x_k, \mathbb{Z}(l))$

FALSE IF $l = p$

BUT GILBERT-MORROW-SCHULZE

$\dim_{\mathbb{R}}(H_{\text{cts}}^m(x_k)(\mathbb{P})) \geq \dim(H_{\text{ét}}^m(x_k, \mathbb{Z}(p))(\mathbb{P}))$



- FINITELY GENERATED $\mathbb{W}(k)$ -MODULE
- CLOSURE RELATION TO DE-RHAM COHOMOLOGY
- p -ADIC COHOMOLOGY THEORY
WELL BEHAVED ONLY FOR SMOOTH
AND PROPER k -VARIETY

Q. IS $H^3_{\text{CTS}}(X|P)$ A (STABLE) -
BIRATIONAL INVARIANT?

WE DON'T KNOW. (OUT YES IF
SOME FORM OF
RESOLUTION OF
SIMULTANEOUS SINGULARITIES)

THM 2 X/k SMOOTH AND PROPER. IF

$\bullet h^{g,i}(X) = h^{i,0}(X) = 0$ AND $\bullet BR(X) = 0$

THEN $H^3_{\text{CTS}}(X|P) = 0$ $\left(\begin{array}{c} \uparrow \\ H^3(X|P) \\ \text{FAT} \end{array} \right)$

RMK

\bullet CONDITIONS ARE SANSFOND IF X/k STABLE RATIONAL

THM 2 SHOWS $H^1_{\text{CTS}} + H^3_{\text{NR}}(\text{FAT})$

$\Rightarrow H^3_{\text{CTS}}(X|P) = 0$

IT SEEMS THAT EACH OF THE TWO ASSUMPTIONS ^{INVARIANT}
IS NOT ENOUGH. $\left\{ \begin{array}{l} \textcircled{1} \text{ EVEN CLASSICAL ENRIQUES} \\ \textcircled{2} \text{ COXETER BUNDLE CONSTRUCTION ?} \end{array} \right.$

AUEL, BIGAZZI,
 BÖHNING, VAN BOMMEYER

(5) PROOF OF THEOREM 2: DIFFERENTIALS
 FORM IN CHARACTERISTIC p .

KEY POINT

$N_S(X) \otimes k$

$\swarrow \mathcal{O}_{CNS} \searrow \mathcal{O}_{DR}$ IS SURJECTIVE!

$$0 \rightarrow H^2_{CNS}(X) \xrightarrow{\substack{\text{res} \\ p}} H^2_{DR}(X) \rightarrow H^2_{BNS}(X)[p] \rightarrow 0$$

RMK (a) IN GENERAL \mathcal{O}_X IS NOT INJECTIVE.

E.G. X SUPER SINGULAR $K3$

$$N_S(X) \otimes k \cong k^{22} \quad H^2_{DR}(X) \cong k^{22}$$

BUT $1 \leq \dim(\ker(\mathcal{O}_X)) \leq 19$

(AND IN + ILLUSIE) WHILE $h^{2,0} = h^{0,2} = 1$

(6) NO THEORY OF RESIDUES AVAILABLE.

E.G. DEX SMOOTH \mathbb{C}^n DIVISOR

EXACT SEQUENCE

$$H^2_{DR}(X) \rightarrow H^2_{DR}(X-D) \rightarrow H^0_{DR}(D)$$

FINITE DIMENSIONAL
INFINITE DIMENSIONAL

TWO STEPS

(1) $\mathcal{O}_{DNS} : \mathcal{N}_S(X) \otimes \mathcal{O}_D \cong \frac{H^2_{DR}(X)}{\mathcal{P}}$

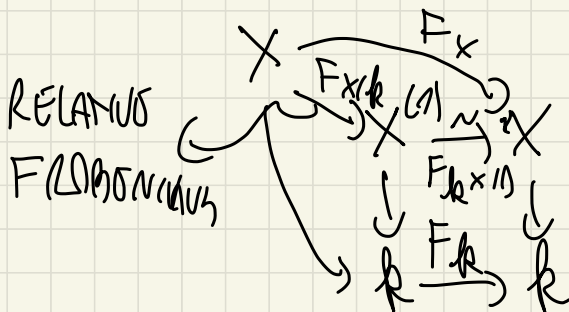
ISG.

(2) $\dim(\mathcal{N}_S(X) \otimes \mathcal{O}_D) = \dim(H^2_{DR}(X))$

FRL: (1) DE-RHAM WIT-COMPLEX (ILLUSIE)

(2) (ITZAS) CANONIC MORPHISMS

I WILL TALK A LITTLE ABOUT (2).



$$\left[\begin{array}{l} X = \text{pt} \quad K[X] \rightarrow K[X^p] = K[X] \\ aX \mapsto aX^p \\ \text{AS SCHEMES } X^{(1)} \simeq X \end{array} \right]$$