REDUCTION MODULO P OF NOETHER'S PROBLEM (WORK IN PROGRESS WITH DOMENICO VALLONI)

① NOETHER'S PROBLEM AND SPECIALISATION

k FIELD, $\mathrm{CHAR}(k) = p \geq 0$, ℓ PRIME NUMBER.
$k = \bar{k}$

<u>DEF</u> • X, Y k-VARIETIES (IRREDUCIBLES)

$X \overset{SB}{\sim} Y$ (X IS STABLY BIRATIONAL TO Y)

IF $\exists\, m, n \in \mathbb{N}_{\geq 0}$ SUCH THAT

$$X \times \mathbb{P}^m \overset{B}{\sim} Y \times \mathbb{P}^n \quad (\overset{B}{\sim} \text{ BIRATIONAL TO})$$

• $SB(k) = \{\text{VARIETIES}/k\} / \overset{SB}{\sim}$

• $X/k \rightsquigarrow [X]_k \in SB(k)$

• X IS STABLY RATIONAL IF $[X]_k = [\bullet]_k$

G FINITE (ℓ-GROUP) $G \leq GL(V)$

V FIN DIMENSIONAL k-VECTOR SPACE

$\left[\frac{V}{G}\right]_k$ DOES NOT DEPEND ON THE CHOICE OF GRV (NO-NAME LEMMA)

NOETHER'S PROBLEM (1917):

IS $\left[\frac{V}{G}\right]_k = [\,\bullet\,]_k$?

ANSWER:

- NO IN GENERAL — EXAMPLES WITH $\ell \neq p$
  - SALTMAN '83 — $|G| = \ell^9$
  - BOGOMOLOV '87 — $|G| = \ell^6$
  - DIFFERENT NATURE — PEYRE '08 — $|G| = \ell^{12}$

- YES SOMETIMES:
  - G ABELIAN, $p \nmid G$ — SWAN '83, FISHER '15.
  - $|G| = p^n$ ($p = \mathrm{CHAR}(k)$) — KUNIYOSHI '59

UPSHOT: GEOMETRY OF V/G DEPENDS ON $\mathrm{CHAR}(k)$.

IN BIRATIONAL GEOMETRY ARE USED SPECIALISATION TECHNIQUES, PUTTING VARIETIES IN FAMILIES AND STUDYING CHANGE OF GEOMETRY (KOLLÁR, TOTARO, VOISIN, HASSETT, COLLIOT-THÉLÈNE, PIRUTKA, SCHREIEDER, ...)

Q: CAN WE INTERPOLATE V/G BETWEEN CHARACTERISTIC ZERO AND p?

Q(G): $\exists$ COMPLETE VAL. RING R OF MIXED CHARACTERISTIC $(0,p)$ $(p>0)$ AND A SMOOTH PROPER R-SCHEME $X \to \mathrm{Spec}(R)$ SUCH THAT $[X_{K}] = [G]_{K}$ AND $[X_{k}] = [G]_{k}$?

$k = \bar{k} = R/m \quad K = \bar{K} =$ FRAC(R)

THM 1 (AV) THE ANSWER TO Q(a) IS NO, $\forall$ GROUPS CONSTRUCTED FROM SALTMAN AND BOGOMOLOV. VAR

RMK BIRATIONAL TYPES CAN VARY IN FAMILIES ← EQUICARA
(STABLY) CUNSIN, SHINDER, NICAISE, SCHREIDERER
MIXED: FREI (TO APPEAR) + HASSETT, TALUANAND

(2) (STABLY) - BIRATIONAL INVARIANTS.

FROM CONG. TO CHARACT. OF SOME CUBIC FOURFOLD

X/k SMOOTH AND PROPER.

EX 1 • $h^{i,0} := \dim(H^0(X, \Omega^i))$ IS A STABLY BIRATIONAL INVARIANT (EASY HANDOUS)

• $h^{0,i}$ $\quad$ " $\quad \begin{cases} = h^{i,0} \text{ IN CHAR } 0. \\ \text{CHATZISTAMATIOU - RÜLLING IN CHAR} > 0 \end{cases}$

RMK $p = 2$ $\quad A \to R$ ABELIAN SURFACE

A_k SUPERSINGULAR

$KUM(A_K) :=$ MINIMAL RESOLUTION OF $A_K/[-1]$ IS A K3 $\quad (H^0(X, \Omega^2) \neq 0)$

∇

- $KUM(A_k) := \quad \shortparallel$ RATIONAL VARIETY

THM (SHIODA '74, KATSURA '99) $(H^0(X, \Omega^2) = 0)$

$\not\exists$ $\mathcal{X} \to R$ SMOOTH PROPER SUCH THAT

$$[X_k] = [KUM(A_k)] \quad \text{AND} \quad [X_K] = [KUM(A_K)]$$

BY SEMICONTINUITY $\dim H^0(X_k, \Omega^2) \geq \dim(H^0(X_K, \Omega^2))$

LAZDA–SKOROBOGATOV: SUCH A MODEL $\mathcal{X} \to R$ EXISTS IF A_k IS NOT SUPERSINGULAR.

EX 2 $\quad AM_\ell(X)$ ARTIN–MUMFORD INVARIANT

$\ell \neq p$ $\quad \dim_{\mathbb{F}_\ell}(H^3_{ET}(X)[\ell]) \quad \left(\left(\frac{Br(X)}{Br(X)_{Div}}\right)[\ell]\right)$

IS A BIRATIONAL INVARIANT.

SALTMAN–BOGOMOLOV EXAMPLES:

$$AM_\ell(V/G) \neq 0 \quad \text{(NOT IN PETRE EXAMPLES)}$$

THM 1' (AV) $X \to R$ SMOOTH AND PROPER

IF X_k IS RATIONAL $H^3(X_{\bar{K}}, \mathbb{Z}_p)[p] = 0$.

(THM 1' $\Rightarrow$ THM 1)

3 INTEGRAL p-ADIC HODGE THEORY

IF $\ell \neq p$ $H^i(X_{\bar{K}}, \mathbb{Z}/\ell) \cong H^i(X_k, \mathbb{Z}/\ell)$

FALSE IF $\ell = p$

BUT BHATT-MORROW-SCHOLZE

$$\dim_k \left(H^n_{\text{crys}}(X_k)[p] \right) \geq \dim \left(H^n_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_p)[p] \right)$$

- FINITELY GENERATED $W(k)$-MODULE
- CLOSELY RELATED TO DE-RHAM COHOMOLOGY
- p-ADIC COHOMOLOGY THEORY WELL BEHAVED ONLY FOR SMOOTH AND PROPER k-VARIETY

Q. IS $H^3_{CNS}(X)[p]$ A (STABLY)-BIRATIONAL INVARIANT?

WE DON'T KNOW. (BUT YES IF STRONG FORM OF RESOLUTION OF SINGULARITIES HOLDS)

THM 2 X/k SMOOTH AND PROPER. IF

- $h^{0,i}(X) = h^{i,0}(X) = 0$ AND
- $Br(X) = 0$ $\left(= H^3_{FLAT}(X, \mathbb{Z}_p(1))_{TORS} \right)$

THEN $H^3_{CNS}(X)[p] = 0$

RMK

- CONDITIONS ARE SATISFIED IF X/k (STABLY RATIONAL)
-

THM 2 SHOWS HYP HODGE + H^3_{NR} (FLAT) $\Rightarrow H^3_{CNS}(X)[p] = 0$

IT SEEMS THAT EACH OF THE TWO ASSUMPTIONS (INDIVIDUALLY) IS NOT ENOUGH. $\left\{ \begin{array}{l} ① \text{ EVEN CLASSICAL ENRIQUES} \\ ② \text{ CONIC BUNDLE CONSTRUCTION BY} \end{array} \right.$?

AVEL, BIGAZZI, BÖHNING, VON BOTHMER

(5) PROOF OF THEOREM 2.: DIFFERENTIALS FORM IN CHARACTERISTIC p.

KEY POINT

$$\begin{tikzcd} & NS(X) \otimes k \arrow[dl, "cl_{crys}"'] \arrow[d, "cl_{DR}"] \\ 0 \to H^2_{crys}(X)/p \arrow[r] & H^2_{DR}(X) \arrow[r] & H^3_{crys}(X)[p] \to 0 \end{tikzcd}$$

cl_{DR} IS SURJECTIVE!

RMK (a) IN GENERAL cl IS NOT INJECTIVE.

E.G. X SUPERSINGULAR K3

$NS(X) \otimes k \cong k^{22}$ $\quad H^2_{DR}(X) \cong k^{22}$

BUT $1 \leq \dim(\ker(cl_{DR}) \leq 10$

(ARTIN + ILLUSIE) WHILE $h^{0,2} = h^{2,0} = 1$

(b) TWO THEORY OF RESIDUES AVAILABLE.

E.G. $D \subseteq X$ SMOOTH (AMPLE [illegible]) DIVISOR

$\exists$ EXACT SEQUENCE $\underbrace{H^2_{DR}(X)}_{\text{FINITE DIMENSIONAL}} \to \underbrace{H^2_{DR}(X-D)}_{\text{INFINITE DIMENSIONAL.}} \to \underbrace{H^0_{DR}(D)}_{\text{FINITE DIMENSIONAL}}$

TWO STEPS

① $c_{CRIS} : NS(X)\otimes k \to H^2_{CRIS}(X)/p$
ISO.

② $\dim(NS(X)\otimes k) = \dim(H^2_{DR}(X))$

FOR: ① DE-RHAM WITT-COMPLEX (ILLUSIE)

② (ITERATED) CARTIER MORPHISM(S)

I WILL TALK A LITTLE ABOUT ②.

$$\begin{tikzcd}
X \arrow[r, "F_{X/k}"] \arrow[rr, bend left, "F_X"] \arrow[dr] & X^{(1)} \arrow[r, "F_k \times 1", "\sim"'] \arrow[d] & X \arrow[d] \\
 & k \arrow[r, "F_k"] & k
\end{tikzcd}$$

RELATIVE FROBENIUS: $F_{X/k}$

$\left[\begin{array}{l} X = \mathbb{A}^1 \quad k[x] \to k[x^p] \subseteq k[x] \\ \qquad\qquad ax \mapsto ax^p \end{array}\right]$

AS SCHEMES $X^{(1)} \simeq X$