Geometry and arithmetic in families of varieties

Emiliano Ambrosi

Oberseminar (MPIM)

24 October 2019

Arithmetic geometry

- k field of characteristic $p \geq 0$ and $Y \subseteq \mathbb{P}^n$ smooth projective k-variety;

Arithmetic geometry

- k field of characteristic $p \geq 0$ and $Y \subseteq \mathbb{P}^n$ smooth projective k-variety;
- Y zero locus of homogeneous polynomials in $k[x_0, ..., x_n]$;

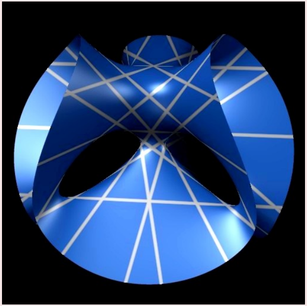
Arithmetic geometry

- k field of characteristic $p \geq 0$ and $Y \subseteq \mathbb{P}^n$ smooth projective k-variety;
- Y zero locus of homogeneous polynomials in $k[x_0, ..., x_n]$;
- $Y(k) := \{(a_1, ..., a_n) \in \mathbb{P}^n_k \,|\, f(a_1, ..., a_n) = 0\}$;

Arithmetic geometry

- k field of characteristic $p \geq 0$ and $Y \subseteq \mathbb{P}^n$ smooth projective k-variety;
- Y zero locus of homogeneous polynomials in $k[x_0, ..., x_n]$;
- $Y(k) := \{(a_1, ..., a_n) \in \mathbb{P}^n_k \, | f(a_1, ..., a_n) = 0\}$;

Cubic surface



We study:

- Arithmetic:

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?
  - Same questions with more exotic arithmetic fields: $k = \mathbb{Q}(T), \mathbb{F}_p, \mathbb{F}_p(T)$.

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?
  - Same questions with more exotic arithmetic fields: $k = \mathbb{Q}(T), \mathbb{F}_p, \mathbb{F}_p(T)$.
- Geometry:

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?
  - Same questions with more exotic arithmetic fields: $k = \mathbb{Q}(T), \mathbb{F}_p, \mathbb{F}_p(T)$.
- Geometry:
  - $k = \mathbb{C}$, what is the topological space $Y(\mathbb{C})$?

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?
  - Same questions with more exotic arithmetic fields: $k = \mathbb{Q}(T), \mathbb{F}_p, \mathbb{F}_p(T)$.
- Geometry:
  - $k = \mathbb{C}$, what is the topological space $Y(\mathbb{C})$?
  - How many lines $Y(\mathbb{C})$ contain?

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?
  - Same questions with more exotic arithmetic fields: $k = \mathbb{Q}(T), \mathbb{F}_p, \mathbb{F}_p(T)$.
- Geometry:
  - $k = \mathbb{C}$, what is the topological space $Y(\mathbb{C})$?
  - How many lines $Y(\mathbb{C})$ contain?
  - Same questions for every algebraically closed field $k = \overline{\mathbb{Q}(T)}, \overline{\mathbb{F}_p}, \overline{\mathbb{F}_p(T)}$.

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?
  - Same questions with more exotic arithmetic fields: $k = \mathbb{Q}(T), \mathbb{F}_p, \mathbb{F}_p(T)$.
- Geometry:
  - $k = \mathbb{C}$, what is the topological space $Y(\mathbb{C})$?
  - How many lines $Y(\mathbb{C})$ contain?
  - Same questions for every algebraically closed field $k = \overline{\mathbb{Q}(T)}, \overline{\mathbb{F}_p}, \overline{\mathbb{F}_p(T)}$.
- Relation between geometry and arithmetic:

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?
  - Same questions with more exotic arithmetic fields: $k = \mathbb{Q}(T), \mathbb{F}_p, \mathbb{F}_p(T)$.
- Geometry:
  - $k = \mathbb{C}$, what is the topological space $Y(\mathbb{C})$?
  - How many lines $Y(\mathbb{C})$ contain?
  - Same questions for every algebraically closed field $k = \overline{\mathbb{Q}(T)}, \overline{\mathbb{F}_p}, \overline{\mathbb{F}_p(T)}$.
- Relation between geometry and arithmetic:
  - $\mathbb{Q} \hookrightarrow \mathbb{C}$: can we relate $Y(\mathbb{Q})$ and $Y(\mathbb{C})$?

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?
  - Same questions with more exotic arithmetic fields: $k = \mathbb{Q}(T), \mathbb{F}_p, \mathbb{F}_p(T)$.
- Geometry:
  - $k = \mathbb{C}$, what is the topological space $Y(\mathbb{C})$?
  - How many lines $Y(\mathbb{C})$ contain?
  - Same questions for every algebraically closed field $k = \overline{\mathbb{Q}(T)}, \overline{\mathbb{F}_p}, \overline{\mathbb{F}_p(T)}$.
- Relation between geometry and arithmetic:
  - $\mathbb{Q} \hookrightarrow \mathbb{C}$: can we relate $Y(\mathbb{Q})$ and $Y(\mathbb{C})$?
  - Theorem (Faltings '83):

    $k = \mathbb{Q}$, Y algebraic curve. If $Y(\mathbb{C})$ is not homeomorphic to a sphere or a torus, then $Y(\mathbb{Q})$ is finite.

We study:

- Arithmetic:
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q}) \neq \emptyset$?
  - $k = \mathbb{Q}$, is $Y(\mathbb{Q})$ finite?
  - Same questions with more exotic arithmetic fields: $k = \mathbb{Q}(T), \mathbb{F}_p, \mathbb{F}_p(T)$.
- Geometry:
  - $k = \mathbb{C}$, what is the topological space $Y(\mathbb{C})$?
  - How many lines $Y(\mathbb{C})$ contain?
  - Same questions for every algebraically closed field $k = \overline{\mathbb{Q}(T)}, \overline{\mathbb{F}_p}, \overline{\mathbb{F}_p(T)}$.
- Relation between geometry and arithmetic:
  - $\mathbb{Q} \hookrightarrow \mathbb{C}$: can we relate $Y(\mathbb{Q})$ and $Y(\mathbb{C})$?
  - Theorem (Faltings '83):

    $k = \mathbb{Q}$, Y algebraic curve. If $Y(\mathbb{C})$ is not homeomorphic to a sphere or a torus, then $Y(\mathbb{Q})$ is finite.
  - $k \hookrightarrow \overline{k}$: can we relate $Y(k)$ and $Y(\overline{k})$?

Linearisation

- Varieties are too complicated, geometry is too complicated, arithmetic is too complicated.

Linearisation

- Varieties are too complicated, geometry is too complicated, arithmetic is too complicated.
- So one can try to linearise them:

Linearisation

- Varieties are too complicated, geometry is too complicated, arithmetic is too complicated.
- So one can try to linearise them:

$$\{\mathit{Varieties}\} \longrightarrow \{\mathit{Vector\ spaces}\}$$

$$Y \longrightarrow V(Y)$$

Linearisation

- Varieties are too complicated, geometry is too complicated, arithmetic is too complicated.
- So one can try to linearise them:

$$\{\mathit{Varieties}\} \longrightarrow \{\mathit{Vector\ spaces}\}$$

$$Y \longrightarrow V(Y)$$

$$\{\mathit{Varieties}\} \longrightarrow \{\mathit{Vector\ spaces} + \mathit{extra\ structure}\}$$

$$Y \longrightarrow \underline{V}(Y)$$

Linearisation

- Varieties are too complicated, geometry is too complicated, arithmetic is too complicated.
- So one can try to linearise them:

$$\{\textit{Varieties}\} \longrightarrow \{\textit{Vector spaces}\}$$

$$Y \longrightarrow V(Y)$$

$$\{\textit{Varieties}\} \longrightarrow \{\textit{Vector spaces} + \textit{extra structure}\}$$

$$Y \longrightarrow \underline{V}(Y)$$

Extra structure

Action of a group, filtration, an automorphism, etc...

Néron Severi group

NS(Y)

- Divisor $Z \subseteq Y$:= codimension 1 subvariety (e.g. curve in surface, surface in threefold)

Néron Severi group

NS(Y)

- Divisor $Z \subseteq Y$:= codimension 1 subvariety (e.g. curve in surface, surface in threefold)
- $NS(Y) \otimes \mathbb{Q} :=$ Néron-Severi group of Y:

Néron Severi group

NS(Y)

- Divisor $Z \subseteq Y$:= codimension 1 subvariety (e.g. curve in surface, surface in threefold)
- $NS(Y) \otimes \mathbb{Q} :=$ Néron-Severi group of Y:
  - $Z^1(Y)$ free $\mathbb{Q}$-vector space on divisors of Y

Néron Severi group

NS(Y)

- Divisor $Z \subseteq Y$:= codimension 1 subvariety (e.g. curve in surface, surface in threefold)
- $NS(Y) \otimes \mathbb{Q} :=$ Néron-Severi group of Y:
    - $Z^1(Y)$ free $\mathbb{Q}$-vector space on divisors of Y
    - equivalence relation $\sim_{num}$: $Z \sim_{num} Z'$ if for every curve $C \subseteq Y$

$$|C \cap Z| = |C \cap Z'|;$$

Néron Severi group

NS(Y)

- Divisor $Z \subseteq Y$:= codimension 1 subvariety (e.g. curve in surface, surface in threefold)
- $NS(Y) \otimes \mathbb{Q} :=$ Néron-Severi group of Y:
  - $Z^1(Y)$ free $\mathbb{Q}$-vector space on divisors of Y
  - equivalence relation $\sim_{num}$: $Z \sim_{num} Z'$ if for every curve $C \subseteq Y$

$$|C \cap Z| = |C \cap Z'|;$$

  - $NS(Y) \otimes \mathbb{Q} := Z^1(Y)/\sim$; finite dimensional $\mathbb{Q}$ vector space

Néron Severi group

Examples

- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq \mathbb{Q}$, generated by an hyperplane;

Néron Severi group

Examples

- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq \mathbb{Q}$, generated by an hyperplane;
- $NS(\mathbb{P}^1 \times \mathbb{P}^1) \otimes \mathbb{Q} \simeq \mathbb{Q} \times \mathbb{Q}$, generated by $\mathbb{P}^1 \times \{*\}$ and $\{*\} \times \mathbb{P}^1$

Néron Severi group

Examples

- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq \mathbb{Q}$, generated by an hyperplane;
- $NS(\mathbb{P}^1 \times \mathbb{P}^1) \otimes \mathbb{Q} \simeq \mathbb{Q} \times \mathbb{Q}$, generated by $\mathbb{P}^1 \times \{*\}$ and $\{*\} \times \mathbb{P}^1$

Base change

- $k \hookrightarrow \overline{k}$, $NS(Y) \otimes \mathbb{Q} \subseteq NS(Y_{\overline{k}}) \otimes \mathbb{Q}$.

Néron Severi group

Examples

- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq \mathbb{Q}$, generated by an hyperplane;
- $NS(\mathbb{P}^1 \times \mathbb{P}^1) \otimes \mathbb{Q} \simeq \mathbb{Q} \times \mathbb{Q}$, generated by $\mathbb{P}^1 \times \{*\}$ and $\{*\} \times \mathbb{P}^1$

Base change

- $k \hookrightarrow \overline{k}$, $NS(Y) \otimes \mathbb{Q} \subseteq NS(Y_{\overline{k}}) \otimes \mathbb{Q}$.
- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq NS(\mathbb{P}^n_{\overline{k}}) \otimes \mathbb{Q} \simeq \mathbb{Q}$

Néron Severi group

Examples

- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq \mathbb{Q}$, generated by an hyperplane;
- $NS(\mathbb{P}^1 \times \mathbb{P}^1) \otimes \mathbb{Q} \simeq \mathbb{Q} \times \mathbb{Q}$, generated by $\mathbb{P}^1 \times \{*\}$ and $\{*\} \times \mathbb{P}^1$

Base change

- $k \hookrightarrow \overline{k}$, $NS(Y) \otimes \mathbb{Q} \subseteq NS(Y_{\overline{k}}) \otimes \mathbb{Q}$.
- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq NS(\mathbb{P}^n_{\overline{k}}) \otimes \mathbb{Q} \simeq \mathbb{Q}$
- $k = \mathbb{Q}$,
  $Y := x^2 + y^2 + z^2 + w^2 = (x + iy)(x - iy) + (z + iw)(z - iw) = 0$

Néron Severi group

Examples

- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq \mathbb{Q}$, generated by an hyperplane;
- $NS(\mathbb{P}^1 \times \mathbb{P}^1) \otimes \mathbb{Q} \simeq \mathbb{Q} \times \mathbb{Q}$, generated by $\mathbb{P}^1 \times \{*\}$ and $\{*\} \times \mathbb{P}^1$

Base change

- $k \hookrightarrow \overline{k}$, $NS(Y) \otimes \mathbb{Q} \subseteq NS(Y_{\overline{k}}) \otimes \mathbb{Q}$.
- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq NS(\mathbb{P}^n_{\overline{k}}) \otimes \mathbb{Q} \simeq \mathbb{Q}$
- $k = \mathbb{Q}$,
  $Y := x^2 + y^2 + z^2 + w^2 = (x + iy)(x - iy) + (z + iw)(z - iw) = 0$
  - $Y_{\overline{k}} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \Rightarrow NS(Y_{\overline{k}}) \otimes \mathbb{Q} \simeq \mathbb{Q}^2$;

Néron Severi group

Examples

- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq \mathbb{Q}$, generated by an hyperplane;
- $NS(\mathbb{P}^1 \times \mathbb{P}^1) \otimes \mathbb{Q} \simeq \mathbb{Q} \times \mathbb{Q}$, generated by $\mathbb{P}^1 \times \{*\}$ and $\{*\} \times \mathbb{P}^1$

Base change

- $k \hookrightarrow \overline{k}$, $NS(Y) \otimes \mathbb{Q} \subseteq NS(Y_{\overline{k}}) \otimes \mathbb{Q}$.
- $NS(\mathbb{P}^n) \otimes \mathbb{Q} \simeq NS(\mathbb{P}^n_{\overline{k}}) \otimes \mathbb{Q} \simeq \mathbb{Q}$
- $k = \mathbb{Q}$,
  $Y := x^2 + y^2 + z^2 + w^2 = (x + iy)(x - iy) + (z + iw)(z - iw) = 0$
  - $Y_{\overline{k}} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \Rightarrow NS(Y_{\overline{k}}) \otimes \mathbb{Q} \simeq \mathbb{Q}^2$;
  - $NS(Y) = \mathbb{Q}$.

Families

$\{Y_x\}_{x \in X}/k$ family of smooth projective varieties $\leftrightarrow$ smooth projective morphism $f : Y \to X$.

Families

$\{Y_x\}_{x \in X}/k$ family of smooth projective varieties $\leftrightarrow$ smooth projective morphism $f : Y \to X$.

Question

How do $NS(Y_x) \otimes \mathbb{Q}$ and $NS(Y_{\overline{x}}) \otimes \mathbb{Q}$ vary with $x \in X$?

Families

$\{Y_x\}_{x \in X}/k$ family of smooth projective varieties $\leftrightarrow$ smooth projective morphism $f : Y \to X$.

Question

How do $NS(Y_x) \otimes \mathbb{Q}$ and $NS(Y_{\overline{x}}) \otimes \mathbb{Q}$ vary with $x \in X$?

$$\begin{array}{ccccc} Y_{\overline{x}} & \longrightarrow & Y_x & \longrightarrow & Y \\ \downarrow & \square & \downarrow & \square & \downarrow f \\ \overline{k(x)} & \longrightarrow & k(x) & \xrightarrow{x} & X \end{array}$$

Families

$\{Y_x\}_{x \in X}/k$ family of smooth projective varieties $\leftrightarrow$ smooth projective morphism $f : Y \to X$, $\eta \in X$ generic point.

Question

How do $NS(Y_x) \otimes \mathbb{Q}$ and $NS(Y_{\overline{x}}) \otimes \mathbb{Q}$ vary with $x \in X$?

$$\begin{array}{ccccc}
Y_{\overline{x}} & \longrightarrow & Y_x & \longrightarrow & Y \\
\downarrow & \square & \downarrow & \square & \downarrow f \\
\overline{k(x)} & \longrightarrow & k(x) & \xrightarrow{x} & X
\end{array}$$

Families

$\{Y_x\}_{x \in X}/k$ family of smooth projective varieties $\leftrightarrow$ smooth projective morphism $f : Y \to X$; $\eta \in X$ generic point.

Question

How do $NS(Y_x) \otimes \mathbb{Q}$ and $NS(Y_{\overline{x}}) \otimes \mathbb{Q}$ vary with $x \in X$?

$$\begin{array}{ccccccccc}
Y_{\overline{x}} & \longrightarrow & Y_x & \longrightarrow & Y & \longleftarrow & Y_\eta & \longleftarrow & Y_{\overline{\eta}} \\
\downarrow & \square & \downarrow & \square & \downarrow{\scriptstyle f} & \square & \downarrow & \square & \downarrow \\
\overline{k(x)} & \longrightarrow & k(x) & \xrightarrow{x} & X & \longleftarrow & k(\eta) & \longleftarrow & \overline{k}(\eta)
\end{array}$$

NS-generic points

$$\begin{array}{ccccc}
Y_{\overline{x}} & \longrightarrow & Y & \longleftarrow & Y_{\overline{\eta}} \\
\downarrow & \square & \downarrow{\scriptstyle f} & \square & \downarrow \\
\overline{k(x)} & \xrightarrow{\overline{x}} & X & \xleftarrow{\overline{\eta}} & \overline{k(\eta)}
\end{array}$$

- Injective specialization morphism:

$$sp_{\eta,x} : NS(Y_{\overline{\eta}}) \otimes \mathbb{Q} \hookrightarrow NS(Y_{\overline{x}}) \otimes \mathbb{Q};$$

NS-generic points

$$\begin{array}{ccccc} Y_{\overline{x}} & \longrightarrow & Y & \longleftarrow & Y_{\overline{\eta}} \\ \downarrow & \square & \downarrow f & \square & \downarrow \\ \overline{k(x)} & \xrightarrow{\overline{x}} & X & \xleftarrow{\overline{\eta}} & \overline{k(\eta)} \end{array}$$

- Injective specialization morphism:

$$sp_{\eta,x} : NS(Y_{\overline{\eta}}) \otimes \mathbb{Q} \hookrightarrow NS(Y_{\overline{x}}) \otimes \mathbb{Q};$$

- Arithmetic variant:

$$sp^{ar}_{\eta,x} : NS(Y_{\eta}) \otimes \mathbb{Q} \hookrightarrow NS(Y_{x}) \otimes \mathbb{Q}.$$

NS-generic points

$$\begin{array}{ccccc} Y_{\overline{x}} & \longrightarrow & Y & \longleftarrow & Y_{\overline{\eta}} \\ \downarrow & \square & \downarrow f & \square & \downarrow \\ \overline{k(x)} & \xrightarrow{\overline{x}} & X & \xleftarrow{\overline{\eta}} & \overline{k(\eta)} \end{array}$$

- Injective specialization morphism:

$$sp_{\eta,x} : NS(Y_{\overline{\eta}}) \otimes \mathbb{Q} \hookrightarrow NS(Y_{\overline{x}}) \otimes \mathbb{Q};$$

- Arithmetic variant:

$$sp^{ar}_{\eta,x} : NS(Y_{\eta}) \otimes \mathbb{Q} \hookrightarrow NS(Y_{x}) \otimes \mathbb{Q}.$$

Definition

x NS-generic (resp. arithmetically NS-generic) if $sp_{\eta,x}$ (resp. $sp^{ar}_{\eta,x}$) isomorphism.

NS-generic points

Questions

- Can we describe the set of (arithmetically) NS-generic closed points?

NS-generic points

Questions

- Can we describe the set of (arithmetically) NS-generic closed points?
- Is the set of (arithmetically) NS-generic closed points not empty?

NS-generic points

Questions

- Can we describe the set of (arithmetically) NS-generic closed points?
- Is the set of (arithmetically) NS-generic closed points not empty?

The answers depend on the arithmetic of k.

NS-generic points

Example 1

- $Y \subseteq \mathbb{P}^n$ of dimension ≥ 3;

NS-generic points

Example 1

- $Y \subseteq \mathbb{P}^n$ of dimension ≥ 3;
- $\{Y_x\}_{x \in X}$ nice pencil hyperplane sections $Y_x \subseteq Y$;

NS-generic points

Example 1

- $Y \subseteq \mathbb{P}^n$ of dimension ≥ 3;
- $\{Y_x\}_{x \in X}$ nice pencil hyperplane sections $Y_x \subseteq Y$;
- x arithmetically NS-generic $\Leftrightarrow NS(Y_x) \otimes \mathbb{Q} \simeq NS(Y) \otimes \mathbb{Q}$.

NS-generic points

Example 1

- $Y \subseteq \mathbb{P}^n$ of dimension ≥ 3;
- $\{Y_x\}_{x \in X}$ nice pencil hyperplane sections $Y_x \subseteq Y$;
- x arithmetically NS-generic $\Leftrightarrow NS(Y_x) \otimes \mathbb{Q} \simeq NS(Y) \otimes \mathbb{Q}$.

Example 1.1

- Veronese's embedding of degree 2

$$\begin{aligned} \mathbb{P}^3_k &\to \mathbb{P}^9_k \\ [x:y:z:w] &\mapsto [x^2:y^2:z^2:w^2:xy:xz:xw:yz:yw:zw]; \end{aligned}$$

NS-generic points

Example 1

- $Y \subseteq \mathbb{P}^n$ of dimension ≥ 3;
- $\{Y_x\}_{x \in X}$ nice pencil hyperplane sections $Y_x \subseteq Y$;
- x arithmetically NS-generic $\Leftrightarrow NS(Y_x) \otimes \mathbb{Q} \simeq NS(Y) \otimes \mathbb{Q}$.

Example 1.1

- Veronese's embedding of degree 2

$$\mathbb{P}^3_k \to \mathbb{P}^9_k$$
$$[x : y : z : w] \mapsto [x^2 : y^2 : z^2 : w^2 : xy : xz : xw : yz : yw : zw];$$

- Hyperplane section $Y_x \leftrightarrow$ Quadric $Q_x \subseteq \mathbb{P}^3$;

NS-generic points

Example 1

- $Y \subseteq \mathbb{P}^n$ of dimension ≥ 3;
- $\{Y_x\}_{x \in X}$ nice pencil hyperplane sections $Y_x \subseteq Y$;
- x arithmetically NS-generic $\Leftrightarrow NS(Y_x) \otimes \mathbb{Q} \simeq NS(Y) \otimes \mathbb{Q}$.

Example 1.1

- Veronese's embedding of degree 2

$$\mathbb{P}^3_k \to \mathbb{P}^9_k$$
$$[x : y : z : w] \mapsto [x^2 : y^2 : z^2 : w^2 : xy : xz : xw : yz : yw : zw];$$

- Hyperplane section $Y_x \leftrightarrow$ Quadric $Q_x \subseteq \mathbb{P}^3$;
- $k = \overline{k} \Rightarrow Q_x \simeq \mathbb{P}^1 \times \mathbb{P}^1$,

$$NS(Q_x) \otimes \mathbb{Q} \simeq \mathbb{Q} \times \mathbb{Q}, \quad \text{while} \quad NS(\mathbb{P}^3) \otimes \mathbb{Q} \simeq \mathbb{Q}.$$

NS-generic points

Theorem 1 (A.)

$p > 0$, k infinite finitely generated (i.e. $k = \mathbb{F}_p(T_1, ..., T_n)$) $\Rightarrow$

NS-generic points

Theorem 1 (A.)

$p > 0$, k infinite finitely generated (i.e. $k = \mathbb{F}_p(T_1, ..., T_n)$) $\Rightarrow$

1. $\exists$ infinitely many arithmetically NS-generic points of bounded degree;

NS-generic points

Theorem 1 (A.)

$p > 0$, k infinite finitely generated (i.e. $k = \mathbb{F}_p(T_1, ..., T_n)$) $\Rightarrow$

1. $\exists$ infinitely many arithmetically NS-generic points of bounded degree;
2. X curve $\Rightarrow$ all but finitely many $x \in X(k)$ NS-generic.

NS-generic points

Theorem 1 (A.)

$p > 0$, k infinite finitely generated (i.e. $k = \mathbb{F}_p(T_1, ..., T_n)$) $\Rightarrow$

1. $\exists$ infinitely many arithmetically NS-generic points of bounded degree;
2. X curve $\Rightarrow$ all but finitely many $x \in X(k)$ NS-generic.

Remark (p=0)

If $p = 0$:

1. is due to André;
2. is due to Cadoret-Tamagawa.

NS-generic points

Problem

$NS(Y_x) \otimes \mathbb{Q}$ difficult to control in general, too geometric.

NS-generic points

Problem

$NS(Y_x) \otimes \mathbb{Q}$ difficult to control in general, too geometric.

Étale cohomology

$$Y \longrightarrow H^2(Y_{\bar{x}}, \mathbb{Q}_\ell(1))$$

NS-generic points

Problem

$NS(Y_x) \otimes \mathbb{Q}$ difficult to control in general, too geometric.

Étale cohomology

$$Y \longrightarrow H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$$

- for $\ell \neq p$ prime $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ finite dimensional $\mathbb{Q}_\ell$-vector space;

NS-generic points

Problem

$NS(Y_x) \otimes \mathbb{Q}$ difficult to control in general, too geometric.

Étale cohomology

$$Y \longrightarrow H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$$

- for $\ell \neq p$ prime $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ finite dimensional $\mathbb{Q}_\ell$-vector space;
- $\pi_1(k(x)) := Gal(\overline{k(x)}, k(x))$ acts (continuously) on $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$;

NS-generic points

Problem

$NS(Y_x) \otimes \mathbb{Q}$ difficult to control in general, too geometric.

Étale cohomology

$$Y \longrightarrow H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$$

- for $\ell \neq p$ prime $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ finite dimensional $\mathbb{Q}_\ell$-vector space;
- $\pi_1(k(x)) := Gal(\overline{k(x)}, k(x))$ acts (continuously) on $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$;
- cycle class map $c_{Y_x} : NS(Y_{\overline{x}}) \otimes \mathbb{Q}_\ell \hookrightarrow H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$.

Tate conjecture

Cycles class map

$$c_Y : NS(Y_x) \otimes \mathbb{Q}_\ell \hookrightarrow H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$$

contained in the fixed points

$$H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))^{\pi_1(k(x))};$$

Tate conjecture

Cycles class map

$$c_Y : NS(Y_x) \otimes \mathbb{Q}_\ell \hookrightarrow H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$$

contained in the fixed points

$$H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))^{\pi_1(k(x))};$$

Conjecture (Tate)

k finitely generated, $\ell \neq p$, then

$$c_Y : NS(Y_x) \otimes \mathbb{Q}_\ell \xrightarrow{\simeq} H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))^{\pi_1(k(x))}.$$

G-generic points

- $\{H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))\}_{x \in X}$ ℓ-adic local system,

G-generic points

- $\{H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))\}_{x \in X}$ ℓ-adic local system,
- e.g. representation of étale fundamental group $\pi_1(X)$.

G-generic points

- $\{H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))\}_{x \in X}$ ℓ-adic local system,
- e.g. representation of étale fundamental group $\pi_1(X)$.

$$\pi_1(k(\eta)) \xrightarrow{\rho_{\ell,\eta}} GL(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$$

$$\pi_1(k(x)) \xrightarrow{\rho_{\ell,x}} GL(H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)))$$

G-generic points

- $\{H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))\}_{x \in X}$ ℓ-adic local system,
- e.g. representation of étale fundamental group $\pi_1(X)$.

$$\begin{array}{ccc}
\pi_1(k(\eta)) & \xrightarrow{\rho_{\ell,\eta}} & GL(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))) \\
\big\downarrow & & \\
\pi_1(X) & & \\
\big\uparrow & & \\
\pi_1(k(x)) & \xrightarrow{\rho_{\ell,x}} & GL(H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)))
\end{array}$$

G-generic points

- $\{H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))\}_{x \in X}$ ℓ-adic local system,
- e.g. representation of étale fundamental group $\pi_1(X)$.

$$\begin{array}{ccc}
\pi_1(k(\eta)) & \xrightarrow{\rho_{\ell,\eta}} & GL(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))) \\
\downarrow & & \vdots \\
\pi_1(X) & & \simeq \\
\uparrow & & \downarrow \\
\pi_1(k(x)) & \xrightarrow{\rho_{\ell,x}} & GL(H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)))
\end{array}$$

G-generic points

- $\{H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))\}_{x \in X}$ ℓ-adic local system,
- e.g. representation of étale fundamental group $\pi_1(X)$.

$$\begin{array}{ccc}
\pi_1(k(\eta)) & \xrightarrow{\rho_{\ell,\eta}} & GL(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))) \\
\downarrow & \overset{\rho_\ell}{\nearrow} & \downarrow \simeq \\
\pi_1(X) & & \\
\uparrow & & \\
\pi_1(k(x)) & \xrightarrow{\rho_{\ell,x}} & GL(H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)))
\end{array}$$

G-generic points

- $\{H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))\}_{x \in X}$ ℓ-adic local system,
- e.g. representation of étale fundamental group $\pi_1(X)$.

$$
\begin{array}{ccc}
\pi_1(k(\eta)) & \xrightarrow{\rho_{\ell,\eta}} & GL(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))) \\
\downarrow & \nearrow^{\rho_\ell} & \downarrow \simeq \\
\pi_1(X) & & \\
\uparrow & & \\
\pi_1(k(x)) & \xrightarrow{\rho_{\ell,x}} & GL(H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)))
\end{array}
$$

Inclusion of ℓ-adic Lie groups

$$\rho_\ell(\pi_1(k(x))) =: \Pi_{\ell,x} \subseteq \Pi_\ell := \rho_\ell(\pi_1(X))$$

Definition

x G-generic (resp. strictly G-generic) if $[\Pi_\ell : \Pi_{\ell,x}] < +\infty$ (resp. $\Pi_{\ell,x} = \Pi_\ell$)

Definition

x G-generic (resp. strictly G-generic) if $[\Pi_\ell : \Pi_{\ell,x}] < +\infty$ (resp. $\Pi_{\ell,x} = \Pi_\ell$)

Proposition (Serre)

k infinite finitely generated $\Rightarrow \exists$ infinitely many strictly G-generic points of bounded degree.

Definition

x G-generic (resp. strictly G-generic) if $[\Pi_\ell : \Pi_{\ell,x}] < +\infty$ (resp. $\Pi_{\ell,x} = \Pi_\ell$)

Proposition (Serre)

k infinite finitely generated $\Rightarrow \exists$ infinitely many strictly G-generic points of bounded degree.

Theorem 2 (A.)

$p > 0$, k finitely generated, X curve $\Rightarrow$ all but finitely many $x \in X(k)$ G-generic.

Definition

x G-generic (resp. strictly G-generic) if $[\Pi_\ell : \Pi_{\ell,x}] < +\infty$ (resp. $\Pi_{\ell,x} = \Pi_\ell$)

Proposition (Serre)

k infinite finitely generated $\Rightarrow \exists$ infinitely many strictly G-generic points of bounded degree.

Theorem 2 (A.)

$p > 0$, k finitely generated, X curve $\Rightarrow$ all but finitely many $x \in X(k)$ G-generic.

Remark (p=0)

If $p = 0$ Theorem 2 is due to Cadoret-Tamagawa.

G-generic vs NS-generic

Tate conjecture predicts:

(Strictly) G-generic points are (arithmetically) NS-generic.

G-generic vs NS-generic

Tate conjecture predicts:

(Strictly) G-generic points are (arithmetically) NS-generic.

Theorem 3 (A.)

$p > 0$, k finitely generated $\Rightarrow$ (strictly) G-generic are (arithmetically) NS-generic.

G-generic vs NS-generic

Tate conjecture predicts:

(Strictly) G-generic points are (arithmetically) NS-generic.

Theorem 3 (A.)

$p > 0$, k finitely generated $\Rightarrow$ (strictly) G-generic are (arithmetically) NS-generic.

Remark (p=0)

If $p = 0$, Theorem 3 due to André.

G-generic vs NS-generic

Tate conjecture predicts:

(Strictly) G-generic points are (arithmetically) NS-generic.

Theorem 3 (A.)

$p > 0$, k finitely generated $\Rightarrow$ (strictly) G-generic are (arithmetically) NS-generic.

Remark (p=0)

If $p = 0$, Theorem 3 due to André.

Conclusion

Existence and abundance of G-generic points (Theorem 2) + Theorem 3 $\Rightarrow$ existence and abundance of NS-generic points (Theorem 1).

THANK YOU FOR THE ATTENTION!