Specialization of representations of the étale fundamental group and applications

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Problem:
Study arithmetic and geometric properties of X_s and $X_{\overline{s}}$ while s is varying in S

- Smooth and proper base change

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\begin{array}{ccc}
\pi_1(k(\eta)) & \xrightarrow{\rho_{\ell,\eta}} & GL(H^i(X_{\overline{\eta}}), \mathbb{Q}_\ell(j)) \\
\downarrow\!\!\downarrow & \overset{\rho_\ell}{\nearrow} & \\
\pi_1(S) & & \downarrow \simeq \\
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- Consider the inclusion

$$\Pi_{\ell,s} \subseteq \Pi_\ell$$

Hilbert's irreducibility theorem (+ Frattini argument)

Proposition (J.P. Serre, T. Terasoma, $\sim$ '85)

There exist infinitely many $s \in |S|$ *such that* $\Pi_{\ell,s} = \Pi_\ell$

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Application

$\rho_{\ell,s}$ *semisimple for all* $s \in |S| \Rightarrow \rho_\ell$ *semisimple.*

Uniform open image theorem

Consider the statement:

$(UOI, X \to S, \leq d)$: $\Pi_{\ell,s}$ is open in Π_ℓ for all but finitely many $s \in S^{\leq d}$ and for all such s,

$$[\Pi_\ell : \Pi_{\ell,s}] \leq C := C(\ell, d, X \to S)$$

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$(UOI, X \to S, \leq d)$ holds if S is a curve and for

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- *(E.A. '16) $p > 0$ and $d = 1$*

Applications

S curve and $p = 0$ (resp $p > 0$)
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- (A.Cadoret, F. Charles '16-E.A.'17) If X_s satisfies Tate conjecture for all $s \in S$ then
$$|Br(X_{\overline{s}})[\ell^\infty]^{\pi_1(k(s))}| \leq C$$
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- For $s \in |S|$, injective map:

$$sp_{\eta,s} : NS(X_{\overline{\eta}}) \otimes \mathbb{Q} \hookrightarrow NS(X_{\overline{s}}) \otimes \mathbb{Q}$$

compatible with

$$H^2(X_{\overline{\eta}}, \mathbb{Q}_\ell(1)) \simeq H^2(X_{\overline{s}}, \mathbb{Q}_\ell(1))$$

and with $ch_{X_{\overline{s}}}$, $ch_{X_{\overline{\eta}}}$

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- *(Y. André '96, A. Cadoret '12) $p = 0$*
- *(E.A.'17) $p > 0$ and $X \to S$ is projective*

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- (A.Cadoret, F. Charles '16-E.A.'17) If S curve and $\overline{\Pi_\ell}^{zar}$ connected then for every $d > 0$ (resp $d = 1$) $\exists C := C(\ell, d, X \to S)$ with

$$[Br(X_{\overline{s}})[\ell^\infty]^{\pi_1(s)} : Br(X_{\overline{\eta}})[\ell^\infty]^{\pi_1(\eta)}] \leq C$$

for all but finitely many $s \in S^{\leq d}$

Main ideas in the proof when $p = 0$

- Variational Hodge conjecture (i.e. Lefschetz theorem on (1,1)-classes + Théorie de Hodge II (P.Deligne '71)) $\Rightarrow$ specialization of $NS(X_{\bar{s}})$ in Betti cohomology controlled via the action of topological fundamental group of S.

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- Comparison between singular and étale cohomology $\Rightarrow$ action studied via the relationship between Π_ℓ and $\Pi_{\ell,s}$

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- 1 is replaced with the variational Tate conjecture in crystalline cohomology (M.Morrow ('14))
- 2 is replaced with the comparison of monodromy groups via the Tannaka formalism. More precisely:

- Instead of Π_ℓ and $\Pi_{\ell,s}$ consider

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- Via independence techniques

$$G^0_\ell = G^0_{\ell,s} \Leftrightarrow G^0_p = G^0_{p,s}$$