UNIVERSITÀ DEGLI STUDI DI MILANO

Facoltà di Scienze e Tecnologia
Dipartimento di Matematica "Federigo Enriques"

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presented by

Emiliano Ambrosi

Tate conjecture for abelian varieties, after Tate, Zarhin, Mori and Faltings

directed by

Anna Cadoret

Fabrizio Andreatta

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Introduction

"The old Marx said:
if you look at the individual capitalist,
you don't understand capitalism.
Look at the whole phenomenon,
in a scientific way, and you'll see it"
Fausto Bertinotti

How much information can we deduce from the cohomology?
Trying to answer to this question is one of the major input for the development of Arithmetic geometry in the last sixty years. Regarding the abelian varieties Tate, in the article [Tat66], proposed the following conjecture:

Over a finitely generated field k, an abelian variety A is uniquely determined, up to isogeny, from the Γ_k-representation $H^1(A, \mathbb{Q}_l)$. Moreover, the representation $H^1(A, \mathbb{Q}_l)$ is semisimple.

The key point is to show that, using the isomorphism $H^1(A, \mathbb{Q}_l)' \to V_l(A)$, the functor V_l is fully faithful. Really we will get something more precise: even the functor T_l is fully faithful.
Tate was able to prove the conjecture over finite fields but his major contribution was to relate the conjecture to a finiteness statement. In particular, he showed that if infinitely many abelian varieties with a polarization of a fixed degree inside a given isogeny class are isomorphic, and some other technical conditions are satisfied, then the conjecture is true. Then he proved that these conditions were satisfied by finite fields.

After a decade, Zarhin trying to prove the conjecture over finitely generated field of positive characteristic, refined the method of Tate, understanding how just the finiteness conditions were enough. Moreover, with his celebrated Zarhin trick, he reduced the problem to the case of principally polarized abelian varieties. In [Zar73b] the conjecture over fields of finite characteristic of transcendence degree 1 was finally proved. The idea to prove the finiteness condition is reminiscent of the proof of Mordell-Weil theorem. Indeed the key step is to use the Northcott's property of the height, not applied to the points of the abelian variety, but directly to the points of the moduli space of abelian varieties. With these ideas in mind, he proved that all the abelian varieties that are involved in the finiteness condition share the same height; a descent argument concludes the proof. One of the key point of Zarhin's proof is to combine some formulas proved by Mumford, in [Mum66] , and the non archimedean inequality (all the valuations in positive characteristic are non archimedean!). The proof over fields with higher transcendence degree can be done in two ways. The first, due to Zarhin, is to replace the role of the height with a sufficient weaker notion. The second, done by Mori in [Mor77], is to do an induction on the transcendence degree and perform a specialization argument.

The most interesting situation of number fields was still unsolved. The interest of the conjecture over number fields was also motived by its link with Shafarevich's conjecture and hence with the Mordell's conjecture. In particular, in [Fal83], Faltings proved the Tate and deduced the other two conjectures from this. The main ingredients of the proof are the same of the proof of Zarhin, in the sense that the idea is again to prove that the height of the points on the moduli space associated to a family of abelian variety is bounded. But, over the number fields, there are some additional complications due to the presence of archimedean valuations. The way to deal with this problem is to introduce the notion of hermitian line bundle and use it to define a new height on an abelian variety. Using a lot of arithmetics, as Neron models, p-divisible groups and the theory of Hodge Tate representations, Faltings was able to bound this height inside a family and then, using some moduli theoretic techniques, to compare it with the modular height in a way that allowed him to bound the last one. For the comparison we will sketch a different pattern, following [DDSMS99].

It is worth mentioning that this conjecture is a particular case of a more general one. Indeed, the general Tate conjecture predicts that, if X is a good variety over a finitely generated field k, the cycle map $Ch^i(X) \otimes \mathbb{Q}_l \to H^{2i}(X, \mathbb{Q}_l)(r)^{\Gamma_k}$ is surjective and that the cohomology groups are semisimple. The relation between the Tate conjecture how stated before and this version, for the H^1 of an abelian variety, depends on the commutativity of the following diagram and some diagram chasing:

$$\begin{tikzcd}
Hom(A,B) \arrow[rr] \arrow[dddd] & & Pic^0(A,B) \arrow[d] \\
& & H^2(A \times B', \mathbb{Q}_l)(1) \arrow[d] \\
& & H^1(A, \mathbb{Q}_l) \otimes H^1(B', \mathbb{Q}_l)(1) \arrow[d] \\
Hom(V_l(A), V_l(B)) & & V_l(A)' \otimes V_l(B) \arrow[ll]
\end{tikzcd}$$

The general Tate conjecture is still widely open. There are just few other cases known, in particular in the last years the conjecture was proved for the $K3$ surfaces over any finitely generated field of characteristic different from 2. The proofs for the $K3$ are inspired by the Deligne's proof of the purity on $K3$ surfaces, using the Kuga–Satake construction that associate to every $K3$ an abelian variety. Using this construction is possible to reduce the conjecture for $K3$ to abelian variety.

The mémoire is organized as follows.
In the first chapter, following a combinations of the work of Zarhin and Tate, we will show how the conjecture is related to some finiteness conditions and we will perform some preliminary reductions.
In the second chapter, we will recall the proof of some general theorems that imply the finiteness condition over finite fields.
In the third chapter, we will prove the so called Zarhin trick and some useful tools about polarizations.
In the fourth chapter, we will prove the conjecture over finitely generated fields of positive characteristic different from 2, following Zarhin and Mori.
In the last chapter, we will prove the conjecture over number fields.
In the appendix, for the convenience of the reader, we recall some general algebraic geometry theorems and some facts about abelian varieties and p-divisible groups.

Chapter 1

Tate conjecture and finiteness conditions

In this section k is any field of characteristic $p \geq 0$, l is a prime different from p and A, B are abelian varieties over k. Define $VarAb_{T_l}$ as the category of abelian varieties with $Hom(A, B) = Hom(A, B) \otimes \mathbb{Z}_l$ and $VarAb_{V_l}$ in the same way. We have two functor $T_l : VarAb_{T_l} \to Rep(\Gamma_k)$ that associate to every abelian variety its Tate module $T_l(A)$ and in the same way we have the functor V_l.

Conjecture 1.0.1 (Tate conjecture)**.**
If k is finitely generated over its prime field, then:
1)T_l is fully faithful
2)$V_l(A)$ is a semisimple representation.

The aim of this thesis is to prove the following:

Theorem 1.0.2. *The Tate conjecture is true.*
1)For finite fields (Tate, [Tat66]).
2)For function fields of positive characteristic different from 2 (Zarhin, [Zar73b] and Mori, [Mor77]).
3)For number fields (Faltings, [Fal83]).

Example. Observe that the conjecture is false when $k = \overline{k}$, with, say, $char(k) = 0$. In fact $V_l(A) \simeq \mathbb{Q}_l^{2g}$, as representation, for every abelian variety of dimension g. It is not hard to show that the Tate conjecture implies that two abelian varieties A, B are isogenous if and only if $V_l(A)$ and $V_l(B)$ are isomorphic as representation. In particular, if the Tate conjecture would be true over k then all the abelian varieties of the same dimension would be isomorphic. And this is clearly not true.

In the next section we will do some reductions that are common to all the proofs.

1.1 Preliminary reductions

The aim of this section is to study the functor $T_l : VarAb_{T_l}(k) \to Rep(\Gamma_k)$. In particular we will show that it is faithful and how the semisimplicity and the fullness are related to some finiteness conditions. We start with a remark that it will be used several times in the sequel.

Remark. It follows from the existence of quotient for subgroups of A that if $f : A \to B$ kills the n torsion, then it is divisible by n in $Hom(A, B)$.

Lemma 1.1.1. *$Hom(A, B)$ is torsion free.*

Proof. To prove that $Hom(A, B)$ is torsion free It is enough to show that the map $Hom(A, B) \to Hom(T_l(A), T_l(B))$ is injective. But if $T_l(f) = 0$ then f is zero on the l^n torsion for every n. Then for every simple abelian sub variety of A, f is zero on a non finite subgroup and hence it is zero. Now theorem A.1.2 implies that f is zero. □

Proposition 1.1.2. *The functor T_l is faithful. In particular $Hom(A, B)$ is free and finitely generated.*

Proof.

- Step 1. For every finitely generated subgroup M of $Hom(A, B)$, $M \otimes \mathbb{Q} \cap Hom(A, B)$ is finitely generated

  Using A.1.2, we can assume A simple and $A = B$. Then, by A.1.3, there exists a polynomial function $\alpha : End(A) \otimes \mathbb{Q} \to \mathbb{Q}$ such that $\alpha(f) = deg(f)$. Since A is simple $deg(f) > 0$ and it is an integer for every $f \in End(A)$. Since $\mathbb{Q} \otimes M$ is finite dimensional and $|\alpha| < 1$ in a neighborhood U of 0 in this space, we get that $U \cap End(X) = 0$. So $End(X) \cap \mathbb{Q} \otimes M$ is discrete in $\mathbb{Q} \otimes M$ and hence finitely generated.

- The map is injective.

  It is enough to show that it is injective on every finitely generated submodule such that $M = M \otimes \mathbb{Q} \cap M$. Suppose that $f_1, ..., f_n$ is a basis of M and that $\sum a_i f_i$ is sent to zero. Choose sequences $n_i(r)$ that converge to a_i. For $r >> 0$ the power of l dividing $n_i(r)$ becomes constant and we denote m the maximum of this powers. But since $T_l(f) = 0$, the power of l dividing $\sum n_i(r) T_l(f)$ is divisible for arbitrary large power of l. In particular $\sum n_i(r) f_i$ kills the l^{m+1} torsion and hence $\sum n_i(r) f_i = l^{n+1} g$ for some g in $\mathbb{Q} \otimes M \cap Hom(A, B) = M$ and this contradicts the previous observation, since the f_i are a basis of M.

∎

Now we reduce the study of the fullness to a study of a $\mathbb{Q}_l$ vector space.

Lemma 1.1.3. *The coker of the map $Hom(A, B) \otimes \mathbb{Z}_l \to Hom(T_l(A), T_l(B))$ is torsion free. In particular, the fullness of the functor T_l is equivalent to the fullness of the functor V_l, thanks to the flatness of $\mathbb{Q}_l$ over $\mathbb{Z}_l$.*

Proof. Suppose that $\psi \in Hom(T_l(A), T_l(B))$ is such that $l^n \psi = T_l(f)$. Then f kills the l^n torsion so that $\phi = l^n g$ for some g and hence $\psi = T_l(g)$. ∎

Lemma 1.1.4. *If the Tate conjecture is true over a finite Galois extension K of k, it is true over k.*

Proof. Suppose that the Tate conjecture is true for every A over K. Then if A is an abelian variety over k we have that $V_l(A)_{\Gamma_K}$ is semisimple by assumption, so that, since Γ_K is of finite index in Γ_k, $V_l(A)$ is semisimple as Γ_k module. For the surjectivity observe that by assumption and Galois descent we have $Hom(A, B) \otimes \mathbb{Q}_l = Hom(A_K, B_K)^{\Gamma_k} \otimes \mathbb{Q}_l = (Hom(A_K, B_K) \otimes \mathbb{Q}_l)^{\Gamma_k} = (Hom(T_l(A), T_l(B))^{\Gamma_K})^{\Gamma_k} = Hom(T_l(A), T_l(B))^{\Gamma_k}$ ∎

Lemma 1.1.5. *It is enough to prove the conjecture for $A = B$.*

Proof. Indeed to prove it for $A \neq B$ it is enough to applied it for $A \times B$, use the decomposition $End(A \times B) = End(A) \times End(B) \times Hom(A, B) \times Hom(B, A)$ and the similar decomposition for $End(V_l(A \times B))$. ∎

Lemma 1.1.6. *$End(A) \otimes \mathbb{Q}_l$ is a semisimple algebra*

Proof. Thanks to A.2.3 it enough to prove that $End(A) \otimes \mathbb{Q}$ is semisimple. By theorem A.1.2 A is isogenous to $A_1^{n_1} \times ... \times A_j^{n_j}$ with the A_i simple and pairwise not isogenous. Observe that $Hom(A_i^{n_i}, A_j^{n_j}) = 0$ if $i \neq j$ and that $Hom(A_i^{n_i}, A_i^{n_i}) = M_{n_i}(End(A_i))$ so that $End(A) \otimes \mathbb{Q} \simeq \prod M_{n_i}(End(A_i)) \otimes \mathbb{Q} = \prod M_{n_i}(End(A_i) \otimes \mathbb{Q})$. To conclude it enough to observe that $End(A_i) \otimes \mathbb{Q}$ is a division algebra, since A_i is simple. ∎

Proposition 1.1.7. *To prove the Tate conjecture it is enough to prove the following assertion: For every abelian variety B, for every Galois stable submodule W of $V_l(B)$, there exists an $u \in End(B) \otimes \mathbb{Q}_l$ such that $uV_l(B) = W$.*

Proof.

- $V_l(A)$ is semisimple. Take a Γ_k invariant submodule W and consider the right ideal $I \subseteq End(A) \otimes \mathbb{Q}_l$ made by those element u such that $u(V_l(A)) \subseteq W$. By 1.1.6 and A.2.2 we get that $I = eEnd(A) \otimes \mathbb{Q}_l$ where e is an idempotent. By hypothesis $eV_l(A) = W$ and hence $V_l(A) = eW \oplus (1 - e)W$. To conclude we just observe that this decomposition is Γ_k invariant since e comes from $End(A) \otimes \mathbb{Q}_l$.

- Clearly $End(A)\otimes\mathbb{Q}_l \subseteq End(V_l(A))^{\Gamma_k}$ so we have only to show the other inclusion. Using 1.1.6 and A.2.1 it is enough to show that $End(V_l(A))^{\Gamma_k} \subseteq Centr_{End(V_l(A))}(Centr_{End(V_l(A))}(End(A)\otimes\mathbb{Q}_l))$ and so we take an element β in the first and a γ in $Centr_{End(V_l(A))}(End(A)\otimes\mathbb{Q}_l)$ and we want to show that $\beta\gamma = \gamma\beta$. The trick is to consider the following space

$$W = \{(x, \beta x) \mid x \in\ T_l(A)\}$$

By the assumption applied to $A\times A$, there exists an $u \in End(A\times A)\otimes\mathbb{Q}_l$ such that $uV_l(A\times A) = W$. Observe that $\begin{bmatrix}\gamma & 0\\ 0 & \gamma\end{bmatrix}$ commutes with u. Then we have:

$$\{(\gamma x, \gamma\beta x) \mid x \in\ T_l(A)\} = \begin{bmatrix}\gamma & 0\\ 0 & \gamma\end{bmatrix} W = \begin{bmatrix}\gamma & 0\\ 0 & \gamma\end{bmatrix} uV_l(A\times A) =$$

$$= u\begin{bmatrix}\gamma & 0\\ 0 & \gamma\end{bmatrix} V_l(A\times A) \subseteq uV_l(A\times A) = W$$

so that $(\gamma x, \gamma\beta x) \in W$ for every x and hence $\gamma\beta x = \beta\gamma x$.

□

We have a polarization $\lambda : A \to A'$ of some degree d.

Proposition 1.1.8 (Zarhin, [Zar73a]). *To prove the Tate conjecture, we can assume in proposition 1.1.7 that W is maximal isotropic with respect of the Weil pairing induced by some polarization.*

Proof. Suppose that we know that for every abelian varieties B and every maximal isotropic Galois invariant subspace M of $V_l(B)$ there exists a $u \in End(B)\otimes\mathbb{Q}_l$ such that $uV_l(B) = M$.

- if $i \in \mathbb{Q}_l$ where i is such that $i^2 = -1$.
  Consider in $V_l(A\times A)$ the subspaces $W_1 = \{(x, ix)|x \in W\}$, $W_2 = \{(x, -ix)|x \in W^{\perp}\}$ and $W_3 = W_1 + W_2$.

  Claim: W_3 is maximal isotropic with respect the pairing induced by $\lambda\times\lambda$

  Proof. 1. W_1 and W_2 are totally isotropic. For example for W_1 we have $e^{\lambda\times\lambda}_{A\times A}((x, ix), (y, iy)) = e^{\lambda}_A(x, y) + e^{\lambda}_A(ix, iy) = e^{\lambda}_A(x, y) + i^2 e^{\lambda}_A(x, y) = 0$ and the same for W_2.
  2. W_1 and W_2 are clearly orthogonal to each other and $W_1 \cap W_2 = 0$
  3. So $Dim(W_3) = Dim(W_1) + Dim(W_2) = \frac{1}{2}Dim(V_l(A\times A))$ (since $W^{\perp}$ has dimension $Dim(V_l(A) - Dim(W))$, since the pairing is non degenerate) and so W_3 is totally isotropic of the maximal dimension and hence maximal isotropic.

  □

  So we have that there exists a $u = \begin{bmatrix}u_{1,1} & u_{1,2}\\ u_{2,1} & u_{2,2}\end{bmatrix} \in M_2(End(A)) = End(A\times A)$ such that $uV_l(A\times A) = W_3$. Now we consider the map $v = (\pi_1 - i\pi_2)\circ u$, where π_i are the canonical projections. The image of this map is W, so that we get two elements $f = u_{1,2} - iu_{2,1}, g = u_{1,2} - iu_{2,2} \in End(A)\otimes\mathbb{Q}_l$ such that $fV_l(A) + gV_l(A) = W$. To conclude we observe that the right ideal of elements v such that $vV_l(A) \subseteq W$ is generated by an idempotent element e, thanks to A.2.2, and that $eV_l(A) = W$

- The proof without the assumption $i \in \mathbb{Q}_l$ is similar, but requires the so called Zarhin Trick. Recall that every positive integer can be written as a sum of four squares thanks to Lagrange's four squares theorem. This implies that in $\mathbb{Q}_l$ we can always write -1 as sum of four squares, say $a^2 + b^2 + c^2 + d^2 = -1$, since a polynomial as a solution in $\mathbb{Z}_l$ in and only if it a solution modulo l^n for every n. Now consider the matrix

$$M = \begin{bmatrix} a & -b & -c & -d\\ b & a & d & -c\\ c & -d & a & b\\ d & c & -b & a\end{bmatrix}$$

  and observe that $M^tM = -Id$, so that this matrix will take care of the absence of $i \in \mathbb{Q}_l$.
  As before define $W_1 = \{(x, Mx)\ x \in W^4\}$, $W_2 = \{(x, -Mx)\ x \in (W^4)^{\perp}\}$ and $W_3 = W_1 + W_2$.

Claim: W_3 is maximal isotropic for the pairing induced by λ^8

Proof. 1. W_1 and W_2 are totally isotropic. For W_1 we have $e^{\lambda^8}_{A^8}((x,Mx),(y,My)) = e^{\lambda^4}_{A^4}(x,y) + e^{\lambda^4}_{A^4}(Mx,My) = e^{\lambda^4}_{A^4}(x,y) + e^{\lambda^4}_{A^4}(M^tMx,y) = 0$ and the same for W_2.

2. W_1 and W_2 are clearly orthogonal to each other and $W_1 \cap W_2 = 0$

3. So $Dim(W_3) = Dim(W_1) + Dim(W_2) = \frac{1}{2}Dim(V_l(A^8))$ (since $W^\perp$ has dimension $Dim(V_l(A) - Dim(W))$ and so W_3 is totally isotropic of the maximal dimension and hence maximal isotropic.

$\square$

As before we get $u_1, ..., u_8 \in End(A) \otimes \mathbb{Q}_l$ such that $u_1V_l(A) + ... + u_8V_l(A) = W$ and hence an element such that $eV_l(A) = W$

$\square$

1.2 Finiteness conditions

So we have to study maximal isotropic Galois invariant subspaces W of $V_l(A)$. Intersecting one on this W with $T_l(A)$ we get an l-divisible subgroup $G = \{G_n := \frac{W \cap T_l(A) + l^nT_l(A)}{l^nT_l(A)}\}$ of $A[l^\infty]$. We now define $B_n = \frac{A}{G_n}$. The multiplication by l^n in A factors trough B_n so that we have isogenies $\psi_n : B_n \to A$ and commutative diagrams:

$$\begin{tikzcd}
T_l(A) \arrow[d] \arrow[dr, "l^n"] & \\
T_l(B_n) \arrow[r, "T_l(\psi_n)"] & T_l(A)
\end{tikzcd}$$

Observe that $Im(T_l(\psi_n)) = W \cap T_l(A) + l^nT_l(A) := X_n$. It is clear that $l^nT_l(A) \subseteq Im(T_l(\psi_n))$ and hence it is enough to prove that $\frac{Im(T_l(\psi_n))}{l^nT_l(A)} = \frac{W \cap T_l(A) + l^nT_l(A)}{l^nT_l(A)}$ and this is the definition of G_n.
The key observation of Tate is the following.

Proposition 1.2.1 (Tate, [Tat66]). *If these B_n fall into finitely many isomorphism classes, the hypothesis of 1.1.7 are satisfied.*

Proof. There exists a B_{n_0} such that we have infinitely many isomorphisms $\alpha_n : B_{n_0} \to B_n$. Consider $u_n = \psi_n \circ \alpha_n \circ \psi_{n_0}^{-1}$, it is an element of $End(A) \otimes \mathbb{Q}_l$ such that $V_l(u_n)$ send in a surjective way (since α_n is an iso) X_{n_0} to X_n.

$$\begin{tikzcd}
B_{n_0} \arrow[r, "\alpha_n"] \arrow[d, "\psi_{n_0}"] & B_n \arrow[d, "\psi_n"] \\
A & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T_l(B_{n_0}) \arrow[r, "T_l(\alpha_n)"] \arrow[d, two heads, "T_l(\psi_{n_0})"] & T_l(B_n) \arrow[d, two heads, "T_l(\psi_n)"] & \\
X_{n_0} \arrow[r, two heads, "V_l(u_n)"] \arrow[d, hook] & X_n \arrow[r, hook] \arrow[d, hook] & X_{n_0} \arrow[dl, hook] \\
V_l(A) \arrow[r, "V_l(u_n)"] & V_l(A) &
\end{tikzcd}$$

Observe that this is a subset of X_{n_0} and hence $V_l(u_n) \in End(X_{n_0}) \subseteq End(V_l(A))$. This is compact , so that there exists a subsequence indexed by some I that is converging to some v. Since the image of $End(A) \otimes \mathbb{Q}_l$ is complete, $v = V_l(u)$ for some $u \in End(A) \otimes \mathbb{Q}_l$. This u will do the work, i.e we will prove that $V_l(u)(V_l(A)) = W$.

- $V_l(u)(X_{n_0}) = \cap_{n \in I} X_n$
  In fact if $x \in X_{n_0}$ $V_l(u)(x) = \lim V_l(u_n)(x) \in \cap_{n \in I} X_n$. If $y \in \cap_{n \in I} X_n$ then for every $i \in I$ we can find x_n such that $V_l(u_n)(x_n) = y$. From x_n we can extract a subsequence that converge to some x, since X_{n_0} is compact. Then $y = lim_n V_l(u_n)(x_n) = V_l(u)(x) \in u(X_{n_0})$.

- $u(V_l(A)) = W$
  Now observe that $\cap_{n \in I} X_n = V_l(A) \cap W$ so that we get

$$V_l(u)(V_l(A)) = \cup_{n \in \mathbb{Z}} V_l(u)(l^n X_{n_0}) = \cup_{n \in \mathbb{Z}} l^n(T_l(A) \cap W) = W$$

□

Before state the main theorem, we make a last important remark. Recall that we have a polarization $\lambda : A \to A'$ of some degree d.

Lemma 1.2.2 (Tate, [Tat66]). *For every n, B_n has a polarization of degree d.*

Proof.
- ψ_n has degree l^{ng}
  It is enough to prove that $\frac{X_n}{l^n T_l(A)}$ has order l^{ng}. For this recall that W, since it is maximal isotropic, it has dimension g and hence $\frac{X_n}{l^n T_l(A)} = \frac{W \cap T_l(A) + l^n T_l(A)}{l^n T_l(A)} = \frac{W \cap T_l(A)}{W \cap l^n T_l(A))} = \frac{W \cap T_l(A)}{l^n (W \cap T_l(A))}$. To conclude observe that $W \cap T_l(A)$ is a free module of rank g.

- The image of the pairing $e_{B_n}^{\lambda_n}$ lies in $l^n \mathbb{Z}_l(1)$
  In fact we have $e_{B_n}^{\lambda_n}(x, y) = e_{B_n}(x, \psi_n' \lambda \psi_n(y)) = e_A^{\lambda}(\psi_n(x), \psi_n(y))$ and so, since the image of ψ_n is X_n, $e_{B_n}^{\lambda_n}(T_l(B_n), T_l(B_n)) \subseteq e_A^{\lambda}(X_n, X_n) \subseteq l^n \mathbb{Z}_l(1)$, where the last equality use the fact that W is totally isotropic.

- Conclusion of the proof.
  We start observing that $\lambda_n := \psi_n' \lambda \psi_n$ is a polarization of B_n of degree $l^{2ng} d$, so we have to produce a polarization $\omega_n : B_n \to (B_n)'$ such that $\lambda_n = \omega_n \circ l^n$ and this happen if and only λ_n kills the l^n torsion. But the previous point tell us that the pairing is zero on the l^n torsion, since the image of the pairing restricted to it lives in $\frac{\mathbb{Z}_l(1)}{l^n \mathbb{Z}_l(1)}$. By the non degeneracy of the pairing, this means that $\lambda_n(x)$ is equal to zero for every $x \in B_n[l^n]$.

□

In conclusion, we can summarize the results of this section in the following theorem:

Theorem 1.2.3. *The Tate conjecture for l is true if one of the following is true for every abelian varieties A with a polarization λ of degree d over k:*
1)There exist finitely many B, up to isomorphism, of the same dimension of A with a polarization of degree d.
2)There exist finitely many B, up to isomorphism, in a given isogeny class that possess a polarization of degree d.
3)For every sub l-divisible group $G = \{G_n\}$ of $A[l^\infty]$ such that every $B_n = \frac{A}{G_n}$ has a polarization of degree d, the B_n fall into finitely many isomorphism classes.

Chapter 2

Proof over finite fields

In this chapter we prove the Tate conjecture over finite fields. The proof has just a little to do with finite fields, since it is a easy consequence of the following three general theorems and theorem 1.2.3.

Theorem 2.0.1. *Let A be an abelian variety over a field k, L a line bundle with associated morphism λ and divisor D. Then we have:*
1)$\chi(L) = \frac{(D)^g}{g!}$
2)$Deg(\lambda) = \chi(L)^2$
3)If L is ample $H^i(A, L) = 0$ for $i > 0$

Theorem 2.0.2. *Let S be a noetherian scheme, W and V two vector bundles of S, π the projection $\mathbb{P}(V) \to S$ and $f \in \mathbb{Q}[x]$. Then the functor*

$$Quot_{\pi^*(W),V,f,S}(T \to S) := \left\{ \begin{array}{c} \textit{surjections } \pi_T^* W \to Q \textit{ with } Q \textit{ locally free and flat over } \mathcal{O}_S \\ \textit{and Hilbert polynomial } f \textit{ on each fiber, up to isomorphism} \end{array} \right\}$$

is a sub functor of $Grass(M)$ for some vector bundle M over S

Theorem 2.0.3. *Let A be an abelian variety over a field k, L an ample line bundle on A. Then L^3 is very ample.*

Proof. See [Mum85] Chapter 17. □

Now we show how this theorems give a proof of the Tate conjecture when k is a finite field using the first point of 1.2.3. We have to show that there exists finitely many isomorphism classes of abelian variety of dimension g with a polarization λ of degree d given by an ample line bundle L. Now for any such abelian variety, thanks to theorem 2.0.3, we have that L^3 is very ample and, thanks to 2.0.1, $H^0(L^3) = \chi(L^3)^3 = \frac{c_1(L^3)^g}{g!} = 3^g \frac{c_1(L)^g}{g!} = 3^g \sqrt{Deg(\lambda)} = 3^g \sqrt{d}$, so that it embeds the variety in $\mathbb{P}^{3^g\sqrt{d}}$. Moreover the Hilbert polynomial of A inside this $\mathbb{P}^n$ is $f(n) = \chi(L^{3n}) = \frac{(c_1(L^{3n}))^g}{g!} = (3n)^g\sqrt{d}$ and so depends only on g and d. This shows that each isomorphism class is a distinct element in $Quot_{k,k^{3^g\sqrt{d}},(3n)^g\sqrt{d},k}(k)$ so it is a distinct k-point a fixed grassmanian. But this is a scheme of finite type over k and so, since k is finite, it has finitely many points.

2.1 Riemann Roch theorem

The key input for the proof of Riemann Roch theorem is the computation of the cohomology of the Poincaré Bundle $\mathcal{P}$ of A. In fact we have the following:

Proposition 2.1.1. *Denote with p_2 the projection $A \times A' \to A'$. We have*

$$R^n p_{2,*}\mathcal{P} = \left\{ \begin{array}{ll} 0 & \textit{if } n \neq g \\ i_0(k) & \textit{if } n = g \end{array} \right. .$$

$$H^n(A \times A', \mathcal{P}) = \left\{ \begin{array}{ll} 0 & \textit{if } n \neq g \\ k & \textit{if } n = g \end{array} \right. .$$

Assuming this proposition we can start with the

Proof. of thm 2.0.1
1)This follows in a trivial way from *A*.3.5 and the fact that $Td(A) = 0$, since the tangent space is free thanks to A.1.15.
2)Assume first that L is non degenerate, i.e $K(L)$ is finite. The trick is to compute $\chi(\Lambda(L))$ in two different way, where $\Lambda(L) = m^*L \otimes p^*L^{-1} \otimes q^*L^{-1} = (Id \times \psi_L)^*\mathcal{P}$ thanks to A.1.7. First observe that we have the following Cartesian diagram:

$$\begin{tikzcd} A \times A \arrow[r, "Id\times\psi_L"] \arrow[d, "q"] & A \times A' \arrow[d, "p_2"] \\ A \arrow[r, "\psi_L"] & A' \end{tikzcd}$$

Thanks to flat base change and the previous proposition we have that

$$R^n q_* \Lambda(L) = R^n q_*(id \times \psi_L)^*\mathcal{P} = \psi_L^*(R^n p_{2,*}\mathcal{P}) = \begin{cases} 0 & \text{if } n \neq g \\ i_*(\mathcal{O}_{K(L)}) & \text{if } n = g \end{cases}$$

where $i : K(L) \to A$ is the inclusion. Since $K(L)$ is finite (and so of zero cohomological dimension) we have that $H^i(A, R^n q_*\Lambda(L)) = 0$ for every $i > 0$ and every n. Now consider the Leray spectral sequence associated to q, i.e $E_2^{u,v} = H^u(A, R^v q_*\Lambda(L)) \Rightarrow H^{u+v}(A \times A, \Lambda(L))$ and observe that, by what we have said so far, in the second page of the spectral sequence all the term with $u \neq 0$ are zero, so that we have an isomorphism $H^u(A \times A, \Lambda(L)) \simeq H^0(A, R^u_{q_*}\Lambda(L))$ for all u, and hence

$$\chi(\Lambda(L)) = (-1)^g Deg(\psi_L)$$

Now we use the second description of $\Lambda(L)$. Observe that $R^i q_*\Lambda(L)$ has support in $K(L)$ and that L is trivial over $K(L)$ (since $K(L)$ is finite over k). Using this remark and the projection formula we get

$$R^n q_*\Lambda(L) = R^n q_*(m^*L \otimes p^*L^{-1} \otimes q^*L^{-1}) = R^n q_*(m^*L \otimes p^*L^{-1}) \otimes L^{-1} = R^n q_*(m^*L \otimes p^*L^{-1})$$

. Observe that the isomorphism of $A \times A$, $m \times p$ send $m^*L \otimes p^*L^{-1}$ to $q^*L \otimes p^*L^{-1}$ so that, using again the degeneracy of the Leray spectral sequence and the Kunneth formula we get

$$H^i(A \times A, \Lambda(L)) = H^i(A \times A, p^*L \otimes q^*L) = \oplus_{u+v=i} H^u(L) \otimes H^v(A, L^{-1})$$

so that, using Poincaré duality and the fact that the tangent bundle is free, we get

$$\chi(\Delta(L)) = \chi(L)\chi(L^{-1}) = (-1)^g \chi(L)^2$$

and hence we are done.
If L is degenerate the argument is similar see [MVdG13] page 133
3)In the previous point we have shown that $H^i(A \times A, \Lambda(L)) = \oplus_{u+v=i} H^u(A, L) \otimes H^v(A, L^{-1})$ and that $H^i(A \times A, \Lambda(L)) \simeq H^0(A, R^i_*\Lambda(L))$. Since just when $i = g$, $h^i(A \times A, \Lambda(L))$ is different from zero and $H^0(X, L)$ is different from zero, we are done. ☐

Before proving the proposition we need a lemma:

Lemma 2.1.2. *If L is a non trivial line bundle then in $Pic^0(A)$ then $H^i(A, L) = 0$ for every i.*

Proof. Since $L \in Pic^0(A)$, $L^{-1} \simeq (-1)^*L$ thanks to A.1.8. So if L has a non zero global section $s : \mathcal{O}_A \to L$ then L^{-1} has a non zero global section $f : \mathcal{O}_A \to L^{-1}$ and this is not possible since $s \otimes L^{-1} \circ f$ would be an automorphism of $\mathcal{O}_A$ so that s would be surjective and hence an isomorphism. So $H^0(A, L) = 0$ and let i the smallest index such that $H^i(A, L) \neq 0$. Then, since $L \in Pic^0(A)$, the choice of i and the Kunneth formula, $H^i(A, m^*L) = H^i(A, p^*L \otimes q^*L) = \oplus_{p+q=i} H^p(A, L) \otimes H^q(A, L) = 0$. But we have that the identity map of A factor trough the multiplication map, so that the identity map of $H^i(A, L)$ factor trough $H^i(A \times A, m^*L) = 0$. ☐

Proof. of prop. 2.1.1

- Step 1. proof for $n \neq g$
  Thanks to the previous lemma and the fact that $\mathcal{P}_{|A \times \{p\}}$ is non trivial and in $Pic^0(X)$ for every $p \in A' - \{(0)\}$, thanks to A.1.8, we have that $H^n(A_p, \mathcal{P}_p) = 0$ so that, by *A*.3.4, $(R^n p_{2,*}\mathcal{P})_p = 0$ for every $0 \neq p \in A'$. We get that $R^n p_{2,*}\mathcal{P}$ is supported at the identity (that has cohomological

dimension 0) and hence $H^i(A', R^n p_{2,*}\mathcal{P}) = 0$ for every $i > 0$. Using again the Leray spectral sequence $H^u(A', R^v p_{2,*}\mathcal{P}) \Rightarrow H^{u+v}(A \times A', \mathcal{P})$, we found that $H^0(A', R^u p_{2,*}\mathcal{P}) = H^u(A \times A', \mathcal{P})$ for every $u \geq 0$. Since p_2 is of relative dimension g, $R^q p_{2,*}\mathcal{P} = 0 = H^u(A \times A', \mathcal{P})$ for $q > g$. Since $\mathcal{P}^{-1} = (-1, 1)\mathcal{P}$, thanks to A.1.8, we get that, using Poincaré duality and the fact that the relative cotangent bundle is free, for every $u < g$ $H^u(A \times A', \mathcal{P}) = H^{2g-u}(A \times A', \mathcal{P}^{-1})' = H^{2g-u}(A \times A', \mathcal{P})' = 0$ and so we are done.

- Step 2. $Hom_{\mathcal{O}_{A\times A'}}(\mathcal{P}, p_2^* G) = Hom_{\mathcal{O}_{A'}}(R^g p_{2,*}\mathcal{P}, G)$
  Theorem A.3.3 give us, for every coherent sheaf G on A' an isomorphism
  $$Hom_{D(A\times A')}(\mathcal{P}[g], p_2^* G[g]) \simeq Hom_{D(A)}(Rp_{2,*}\mathcal{P}[g], G)$$
  . Now the left hand side is nothing else that $Hom_{\mathcal{O}_{A,A'}}(\mathcal{P}, p_2^* G)$ while the right hand side is $Hom_{\mathcal{O}'_A}(R^g p_{2,*}\mathcal{P}, G)$ since, by what we have proved before, $R^g p_{2,*}$ has at most one term different from zero in degree g.

- Step 3. Conclusion of the proof.
  Recall that $R^g p_{2,*}\mathcal{P}$ has support in $0 \in A'$. Now, thanks to theorem A.3.4 and the first point, we have that $R^g p_{2,*}\mathcal{P} \otimes k(0) \simeq H^g(A \times \{0\}, \mathcal{P}_{|A\times 0}) \simeq H^g(A, \mathcal{O}_A) = k$ so that $(R^g p_{2,*})_{(0)} \simeq \frac{(\mathcal{O}_{A'})_{(0)}}{\mathfrak{a}}$ for some ideal $\mathfrak{a} \subseteq \mathfrak{m}$, the maximal ideal of $(\mathcal{O}_{A'})_{(0)}$. We have to show that $\mathfrak{a} = \mathfrak{m}$. If we denote $A(I) = A \times \frac{(\mathcal{O}_{A'})_{(0)}}{I}$ for every ideal $I \subseteq \mathfrak{m}$, we get the following commutative diagram, thanks to the point 2 and the usual adjunctions, in which all the horizontal arrows are isomorphisms:
  $$\begin{tikzcd}
  Hom_{A(\mathfrak{a})}(\mathcal{P}_{|A(\mathfrak{a})}, \mathcal{O}_{A(\mathfrak{a})}) \arrow[r] \arrow[d] & Hom_{A\times A'}(\mathcal{P}, \mathcal{O}_{A(\mathfrak{a})}) \arrow[r] \arrow[d] & Hom_{A'}(\frac{(\mathcal{O}_{A'})_{(0)}}{\mathfrak{a}}, \frac{(\mathcal{O}_{A'})_{(0)}}{\mathfrak{a}}) \arrow[r] \arrow[d] & \frac{A}{\mathfrak{a}} \arrow[d] \\
  Hom_{A(\mathfrak{m})}(\mathcal{P}_{|A(\mathfrak{m})}, \mathcal{O}_{A(\mathfrak{m})}) \arrow[r] & Hom_{A\times A'}(\mathcal{P}, \mathcal{O}_{A(\mathfrak{m})}) \arrow[r] & HomA'(\frac{(\mathcal{O}_{A'})_{(0)}}{\mathfrak{m}}, \frac{(\mathcal{O}_{A'})_{(0)}}{\mathfrak{m}}) \arrow[r] & \frac{A}{\mathfrak{m}} = k
  \end{tikzcd}$$
  The commutativity implies that the first vertical map is surjective, so that we can lift the isomorphism $\mathcal{P}_{|A(\mathfrak{m})} \simeq \mathcal{O}_{A(\mathfrak{m})}$ to a map $\mathcal{P}_{|A(\mathfrak{a})} \simeq \mathcal{O}_{A(\mathfrak{a})}$. Via Nakayama, this map is surjective and hence an isomorphism. But this means, thanks to the universality of $\mathcal{P}$, that we have a map $A(\mathfrak{a}) \to A(\mathfrak{m})$ lying over the natural inclusion and hence $\mathfrak{m} \subseteq \mathfrak{a}$. □

2.2 Quot functor

This section is devoted to the proof of theorem 2.0.2. We want to underline that it is possible to prove a more general theorem, that states the representability of the Quot functor by a locally closed subscheme of a grassmanian. We don't need this, since we are only interested in the finiteness statement. The proof of the representability is based on our construction of an injection of the *Quot* functor in some grassmanian, but then it requires some more work, since it is necessary to show that this injection is representable. So, for our purpose, it is enough to do "half" of the proof of the representability. For the complete proof we refer the interested reader to [Fa05]
We will need a form of a uniform vanishing to make, in an uniform way and using theorem A.3.4, our flat sheaves locally free.

Proposition 2.2.1. *Fix a rational polynomial with integer values $f(t)$ and two natural number p, n. There exists a positive integer $m = m(f, n, p)$ with the following property:
For every field k, for every coherent sheaf $\mathcal{F}$ of $\mathbb{P}^n$ which is a sub sheaf of $\mathcal{O}^p_{\mathbb{P}^n}$ with Hilbert polynomial f, $\mathcal{F}(r)$ is generated by global section and $H^i(\mathcal{F}(r)) = 0$ for $i > 0$ and every $r \geq m$.*

The proof is quite technical so we postpone it. Let us show how this implies our theorem.

Proof. of thm 2.0.2

- Step 1. Use of uniform vanishing.
  Fix a S scheme T, a coherent quotient $\pi_T^* W \to \mathcal{F}$, where $\mathcal{F}$ is a coherent sheaf in $\mathbb{P}^n_T$ flat over $\mathcal{O}_S$ and with Hilbert polynomial f. Denote the kernel of the map $\mathcal{G}$. On each fiber $s \to S$ we have an exact sequence $0 \to \mathcal{G}_s \to \mathcal{O}^p_{\mathbb{P}^n} \to F_s \to 0$ where $n = rank(V)$ and $p = rank(W)$. The last two terms have a fixed Hilbert polynomial. Thanks to the previous proposition there

exists an r, depending only of W, V and f, such that $H^i(T_s, \mathcal{G}_s(r)) = H^i(T_s, (\pi_T^* W)_s(r)) = 0$ and $\mathcal{G}(r)_s, (\pi_T^* W(R))_s$ are generated by global section. Thanks to the exact sequence the same is true for $\mathcal{F}_s(r)$.

- Step 2. Construction of the map.
  Observe that, thanks to A.3.4 and $H^i(T_s, \mathcal{G}_s(r)) = 0$, $R^i\pi_{T,*}\mathcal{G}(r) = 0$ for $i > 0$ so that we have a surjection $\pi_{T,*} pi_T^* W \to \pi_{T,*}\mathcal{F}$. Moreover, since $H^i(T_s, (\pi_T^* W)_s(r)) = H^i(T_s, \mathcal{F}(r)) = 0$ and again theorem A.3.4, $\pi_{T,*}\pi_T^* W$ and $\pi_{T,*}\mathcal{F}$ are locally free. Observe that $Rank(\pi_{T,*}\mathcal{F})(r) = f(r)$, since all the higher cohomology of the fibers are zero. Now $\pi_{T,*}\pi_T^* W(r) = W \otimes_{\mathcal{O}_S} Sym^r(V)$, so that $\pi_{T,*}\pi_T^* W(r) \to \pi_{T,*}\mathcal{F}(r) \in Grass(W \otimes_{\mathcal{O}_S} Sym^r(V), f(r))(T)$. So, since everything depends only on V, W, f we get a map $Quot \to Grass(W \otimes_{\mathcal{O}_S} Sym^r(V), f(r))$, that send, for every S scheme T, a surjection $\pi_T^* W \to \mathcal{F}$ in $\pi_{T,*}\pi_T^* W(r) \to \pi_{T,*}\mathcal{F}(r)$.

- Step 3. The map is injective.
  We have to show that if we know $\pi_{T,*}\pi_T^* W(r) \to \pi_{T,*}\mathcal{F}(r)$, and hence $\pi_{T,*}\mathcal{G}(r)$ as kernel of the map, we can reconstruct $\pi_T^* W \to \mathcal{F}$. For this observe that we have the following commutative diagram with exact rows and surjective vertical maps (since $\pi_T^* W$ and $\mathcal{G}(r)$ are generated by global sections thanks to A.3.4 and so the counits are surjective):

$$\begin{tikzcd}
\pi_T^*\pi_{T,*}\mathcal{G}(r) \arrow[r] \arrow[d] & \pi_T^*\pi_{T,*}\pi_T^* W(r) \arrow[r] \arrow[d] & \pi_T^*\pi_{T,*}\mathcal{F}(r) \arrow[r] \arrow[d] & 0 \\
\mathcal{G}(r) \arrow[r] & \pi_T^* W(r) \arrow[r] & \mathcal{F}(r) \arrow[r] & 0
\end{tikzcd}$$

  So we get that $(\pi_T^* W(r) \to \mathcal{F}(r)) = Cocker(\mathcal{G}(r) \to \pi_T^* W(r)) = Cocker(\pi_T^*\pi_{T,*}\mathcal{G}(r) \to \pi_T^* W(r))$ and hence we can recover $\pi_T^* W(r) \to \mathcal{F}(r)$, since this map is the cokernel of the push via π_T^* of $\pi_{T,*}\mathcal{G}(r) \to \pi_{T,*}\pi_T^* W(r))$ composed with the counit. So we can recover $\pi_T^* W \to \pi_{T,*}\mathcal{F}$ just twisting.

Now we have to prove the uniform vanishing. The idea is to construct a well behaved notion that will allow us to do induction. We fix a field k and we start with a definition.

Definition 2.2.2. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is called m-regular, $m \in \mathbb{N}$ if for every $r \geq m$ we have $H^i(\mathcal{F}(r-i)) = 0$.

Remark. If $\mathcal{F}$ is m-regular and H is an hyperplane in $\mathbb{P}^n$ that does not contain anyone of the (finite) associated points of $\mathcal{F}$ we have that $F_{|H}$ is again m-regular over $\mathbb{P}^{n-1}$, since we have an exact sequence $0 \to F(r-1) \to F(r) \to F_{|H} \to 0$ (locally the multiplication by x_n is injective, where $H : x_n = 0$.)

This is exactly the kind of sheaves that we are looking for thanks to the following lemma.

Lemma 2.2.3. *If $\mathcal{F}$ is m-regular, then:*
1)$H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ for every $r \geq m - i$ and $i \geq 1$
2)$\mathcal{F}(r)$ is generated by global section for every $r \geq m$

Proof. Thanks to flat base change we can assume that k is a infinite field. Then we can choose an hyperplane that does not contain any of the associated point of $\mathcal{F}$, so that we have an exact sequence $0 \to F(r-1) \to F(r) \to F_{|H} \to 0$.
1)We do induction on n and $r = m - i$. For $n = 0$ it's obvious for every r. For $r = m - i$ it is just the definition of m-regularity. Now we just use the exact sequence above, noting that $H^i(\mathbb{P}^n, \mathcal{F}(r-1)) = 0$ by the inductive hypothesis on r and $H^i(H, \mathcal{F}_{|H}) = 0$ thanks to the inductive hypothesis on n.
2)Since for r big enough $H^i(\mathbb{P}^n, \mathcal{F}(r))$ is generated by global section, it is enough to prove that the map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^0(\mathbb{P}^n, \mathcal{F}(r+1))$ is surjective and use induction on r.
Again we do induction on n, since for $n = 0$ is clear. We have the following commutative diagram with exact rows (thanks to the first point and the fact that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(H, \mathcal{O}_H)$ is always surjective)

$$\begin{tikzcd}
 & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{F}(r)) \arrow[r] \arrow[d, "h"] & H^0(H, \mathcal{O}_H) \otimes H^0(H, \mathcal{F}_{|H}(r)) \arrow[r] \arrow[d] & 0 \\
H^0(\mathbb{P}^n, \mathcal{F}(r)) \arrow[r, "f"] & H^0(\mathbb{P}^n, \mathcal{F}(r+1)) \arrow[r, "g"] & H^0(H, F_{|H}(r+1)) &
\end{tikzcd}$$

By induction the right vertical map is surjective, so we get that g is surjective and so $H^0(\mathbb{P}^n, \mathcal{F}(r+1)) = Ker(g) + Im(h) = Im(f) + Im(h)$. But the $Im(f) \subseteq Im(h)$ and we are done. $\square$

So now we can find the uniform vanishing using the following proposition.

Proof. of prop. 2.2.1 Thanks to the previous lemma it is enough to show that $\mathcal{F}$ is m regular for some m that depends only on p, n, f. We do induction on n since $n = 0$ is clear for every polynomial f. Again we can assume that k is a infinite field. Then we can choose an hyperplane that does not contain any of the associated point of $\mathcal{F}$ and of $\frac{\mathcal{O}^p_{\mathbb{P}^n}}{\mathcal{F}}$, so that we have exact sequences $0 \to F(r-1) \to F(r) \to F_{|H} \to 0$ and $0 \to \mathcal{F}_{|H} \to \mathcal{O}^p_H$. Observe that thanks to the first exact sequence, the Hilbert polynomial of $\mathcal{F}_{|H}$ depends only on the Hilbert polynomial of $\mathcal{F}$ so that, by induction and using the second exact sequence, $\mathcal{F}_{|H}$ is m_0 regular for some m_0 that depends only on p, n, f. Now if $r \geq m_0$ and $i > 1$ we have an exact sequences

$$0 \to H^0(\mathbb{P}^n, \mathcal{F}(r-1)) \to H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^0(H, F_{|H}(r)) \to H^1(\mathbb{P}^n, \mathcal{F}(r-1)) \to H^1(\mathbb{P}^n, \mathcal{F}(r)) \to 0$$

$$0 \to H^i(\mathbb{P}^n, \mathcal{F}(r-1)) \to H^i(\mathbb{P}^n, \mathcal{F}(r)) \to 0$$

For r big enough and $i > 1$ $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ and so, thanks to the second exact sequence, get that $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ for every $r \geq m_0$ and $i > 1$.
So we have just to take care of $i = 1$. We have to show that $H^1(\mathbb{P}^n, \mathcal{F}(r))$ vanishes for all $r \geq m$ for some m that depends only on n, p, f. This follows from the following two claim.

- $h^1(\mathbb{P}^n, \mathcal{F}(r))$ is strictly decreasing as function on $r \geq m_0$ until it becomes zero.
  The first exact sequence implies that $h^1(\mathbb{P}^n, \mathcal{F}(r))$ is decreasing and that $h^1(\mathbb{P}^n, \mathcal{F}(r)) = h^1(\mathbb{P}^n, \mathcal{F}(r-1))$ if and only if the map $H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^0(H, F_{|H}(r))$ is surjective. Now look at the diagram of the proof of the previous lemma. The reasoning done there implies that if $H^0(\mathbb{P}^n, \mathcal{F}(r)) \to H^0(H, F_{|H}(r))$ is surjective then it is surjective for every $r' \geq r$ and hence the map $H^1(\mathbb{P}^n, \mathcal{F}(r')) \to H^1(\mathbb{P}^n, \mathcal{F}(r'))$ is an isomorphism for $r' \geq r$. But for r' big enough $H^1(\mathbb{P}^n, \mathcal{F}(r')) = 0$ and hence $h^1(\mathbb{P}^n, \mathcal{F}(r)) = 0$

- $h^1(\mathbb{P}^n, \mathcal{F}(m_0))$ is bounded by a constant that depends only on n, p, f.
  Observe that $h^1(\mathbb{P}^n . \mathcal{F}(m_0)) = h^0(\mathbb{P}^n, \mathcal{F}(m_0)) - \chi(\mathcal{F}(m_o))$ since the higher cohomology group are zero. So we get $h^1(\mathbb{P}^n . \mathcal{F}(m_0)) = h^0(\mathbb{P}^n, \mathcal{F}(m_0)) - f(m_0)$. Now we use the hypothesis that $\mathcal{F} \subseteq \mathcal{O}^p_{\mathbb{P}^n}$ to get an inclusion $\mathcal{F}(m_0) \subseteq \mathcal{O}^p_{\mathbb{P}^n}(m_0)$ and hence that

  $$h^0(\mathbb{P}^n, \mathcal{F}(m_0)) \leq p h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m_0)) = p\binom{n+m_0}{n}$$

  (homogeneous polynomial of degree m_0 in $n+1$ variables). So we have that

  $$h^1(\mathbb{P}^n . \mathcal{F}(m_0)) \leq p\binom{n+m_0}{n} - f(m_0)$$

  and we conclude noticing that all this term depends only on f, p, n.

$\square$

Chapter 3

Interlude: polarizations and theta groups.

In the future we will have to deal with two problems. The first is in the proof of the Tate conjecture over function field, where we will need some condition to avoid the non separable polarizations. In fact, the theory of moduli spaces for abelian variety with a polarization of fixed degree works well when the characteristic of the field does not divide the degree of the polarization. But it could be that every polarization on an abelian variety is divisible by the characteristic of the field, so we will need some trick to change our abelian variety.

Example.

In characteristic zero we will need to do some explicit computation on the moduli space and these computation can be done well over the moduli space of principally polarized abelian variety. So we will need to change our abelian variety with a principal polarized one.

3.1 Lifting of Polarizations

The key will be a very explicit description of the pairing induced by an isogeny. We will need the notion of a sheaf with an action of a group, for the definition and the main theorem of descent see [MVdG13], page 98, chapter 7.

Suppose that $f : A \to B$ is an isogeny between abelian varieties. Then we have that $B \simeq \frac{A}{ker(f)}$, so that f^* induces a bijection between line bundles on B and line bundles on A with an action of $ker(f)$. Suppose that $L \in Pic(B)$ is such that $f^*L \simeq \mathcal{O}_A$. Then L can be seen as an action of $ker(f)$ on the trivial line bundle on A. With a functorial point of view, we take a scheme T over k. Then we have that a line bundle over B_T which becomes trivial when pulled back on $A \times T$ is a morphism $ker(F)(T) \to Aut(\mathbb{A}^1_{A_T}) = \mathcal{O}^*_{A_T} = \mathcal{O}^*_T$.

So we summarize this discussion in the following:

Lemma 3.1.1. *$Ker(f)'(T) = Hom(Ker(f)(T), \mathcal{O}^*_T) = \{ L \in Pic(B_T)$ such that $f^*L \simeq \mathcal{O}_{A_T} \}$*

Remark. One should check that the morphism does not depend on the choice of the isomorphism. See [MVdG13] Proposition 7.4.

Observe now what is $Ker(f')$. For every scheme T over k, $Ker(f')(T)$ is a subset of $B'(T) = \frac{Pic(B_T)}{\pi^*_T(Pic(T))}$. An element L in $B'(T)$ is in $ker(f')(T)$ if and only if it is equivalent to one such that f^*L is equivalent to $\mathcal{O}_{A_T}$, by the very definition of f'. But every such L is uniquely represented by a line bundle such that $f^*L = \mathcal{O}_{A_T}$. Indeed, if $L \in ker(f')(T)$ then $f^*L \simeq \pi^*_T M$ for some M and hence $L' = L \otimes \pi^*_T M^{-1}$ does the job. Conversely, if $L_1 = L_2 \otimes \pi^*_T M$ and $f^*L_1 = f^*L_2$ then $L_1 = L_2$ since $\pi^*_T M = \mathcal{O}_{A_T}$. So we have found:

Lemma 3.1.2. *$Ker(f')(T) = \{ L \in Pic(Y_T)$ such that $f^*L \simeq \mathcal{O}_X \}$*
In particular $Ker(f') = Ker(f)'$.

So we have made very explicit our perfect pairing $Ker(f) \times Ker(f') \to \mathbb{G}_m$. How does it works con T point? Well, we take a couple (x, y). y is a line bundle such that $f^*L = \mathcal{O}_{A_T}$ and hence it is an action of $Ker(f)(T)$ over the trivial line bundle and so it a morphism $\psi_y : Ker(f)(T) \to \mathbb{G}_m(T)$. Then the

pairing send (x, y) to $\psi_y(x) = e_f(x, y)$. Geometrically the action of x send $(a, b, c) \in A \times T \times \mathbb{A}^1$ to $(a + x, b, \psi_y(x)c)$.
We are now ready to find some criterion to lift polarization. Consider the following diagram:

$$A \times A' \xleftarrow[id \times f']{} A \times B' \xrightarrow{f \times Id} B \times B'$$

We have the Poincaré bundles $\mathcal{P}_A$ and $\mathcal{P}_B$ and we have that, by definition of f', $(f \times Id)^*\mathcal{P}_B \simeq (Id \times f')^*\mathcal{P}_A := L$ in a canonical way. Then, since $A \times A' = \frac{A \times B'}{\{0\} \times Ker(f')}$, L has an action of $\{0\} \times Ker(f')$. We have $L_{|A \times Ker(f')} = (Id \times f')^*((\mathcal{P}_A)_{A \times \{0\}}) = \mathcal{O}_{A \times ker(f)}$ in a natural way, see A.1.7. So the action of $q \in Ker(f')$ restricted to $A \times Ker(f')$ is the trivial one, i.e on T point it send (x, y, z) to $(x, y + q, z)$. On the other hand, we have also an action of $\{0\} \times Ker(f)$ on L. And when we restrict this action on $A \times Ker(f')$ we get, applying what said before with $T = ker(f')$, that $q \in Ker(f)$ send (x, y, z) to $(x + q, y, e_f(x, y)a)$. In particular we get that the action of $x \in Ker(f)$ and $y \in Ker(f')$ commute in $A \times Ker(f')$, and hence everywhere since the action can differ just for a constant, if and only if $e_f(x, y) = 1$.

Proposition 3.1.3. *Fix a polarization λ on A and an isogeny $g : A \to C$. Then there exists a polarization η on C such that $g^*\eta = \lambda$ if and only if $ker(f)$ is contained in $ker(\lambda)$ and $ker(f)$ is totally isotropic for the pairing associated to λ.*

Proof. The proof is an application of the discussion above applied with $B = A'$ and $f = \lambda$. A polarization is a line bundle of $A \times A$ and we want to know when it descend to a line bundle on $C \times C = \frac{A}{ker(g)} \times \frac{A}{ker(g)}$. The line bundle descent to $C \times C$ if and only if there is an action of $ker(g) \times ker(g)$. But this is made by two compatible actions of $0 \times ker(f)$ and $ker(f) \times 0$. As we said before the two actions are compatible if and only if $ker(g)$ is totally isotropic for the pairing. □

Corollary 3.1.4. *(Zarhin) $A^4 \times A'^4$ is principally polarized.*

Proof. Suppose that $\lambda : B \to B'$ is a polarization of an abelian variety and α an endomorphism of B. Consider the isogeny f given by $(x, y) \mapsto (x - \alpha(y), \lambda(y))$ and observe that it has the same degree of λ and that $Ker(f) = \{(\alpha(y), y) \mid y \in Ker(\lambda)\}$. Observe that if the polarization $\lambda \times \lambda$ descend to $B \times B'$ then it is principal by a degree computation. By the previous proposition, and unraveling the definition, $\lambda \times \lambda$ descend if $\alpha(ker(\lambda)) \subseteq Ker(\lambda)$ and $e_\lambda(\alpha(x), \alpha(y))e_\lambda(x, y) = 1$ for every $x, y \in Ker(\lambda)$. The first condition is satisfied if $\alpha \circ \lambda = \lambda \circ \alpha$ and under this hypothesis the second condition is satisfied if $e_\lambda(x, (1 + \alpha'\alpha)(y)) = 1$ for $x, y \in Ker(\lambda)$, since

$$e_\lambda(\alpha(x), \alpha(y)) = e_{\lambda \circ \alpha}(x, \alpha(y)) = e_{\alpha \circ \lambda}(x, \alpha(y)) = e_\lambda(x, \alpha'\alpha(y))$$

.
Now take $B = A^4$, choose m such that $ker(\lambda) \subseteq A[m]$, write $m - 1 = a^2 + b^2 + c^2 + d^2$ thanks to the Lagrange for squares theorem and consider the endomorphism

$$\alpha = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

□

Proposition 3.1.5 ([LOZ96]). *The set of abelian sub varieties of A up to isomorphism is finite*

Proof. We prove something more general, namely that the set T of abelian sub varieties of A up to the action of $Aut(A)$ is finite. Define V as set set of right ideals of $End(A) \otimes \mathbb{Q}$ modulo the action of $Aut(A)$. We will construct an injective map $T \to V$ and then we will show that T is finite.

- We define the map $T \to V$ that send Y to $I(Y) \otimes \mathbb{Q}$ where $I(Y) := \{u \in End(A)$ such that $u(A) \subseteq Y\}$. It is clearly well defined, and, to show that it is injective, suppose that $I(Y) \otimes \mathbb{Q} = uI(Z) \otimes \mathbb{Q}$ for some $u \in Aut(A)$. Since $uI(Z) = I(u(Z))$ we can assume $u = 1$. Thanks to A.1.2 there exists a $W \subseteq A$ and an isogeny $Y \times W \to A$, so that there exist a surjective map $\psi : A \to Y$. This implies $\psi(A) = Y$ and hence $\psi \in I(Y) \otimes \mathbb{Q} = I(Z) \otimes Q$ and hence there exists an n such that $n\psi \in I(Z)$. Recalling that the multiplication by n is an isogeny we get $Y = nY = n\psi(A) \subseteq Z$ and by a symmetric reasoning $Z \subseteq Y$.

- We show something more general namely that if F is a semisimple finite dimensional $\mathbb{Q}$ algebra, L is a $\mathbb{Z}$-lattice inside F, G is the set of automorphisms of F as F module, i.e. the invertible elements of F, such that $\sigma(L)=L$, then the set of right ideals of F modulo the natural action of G is finite. Then we will apply $F=End(A)\otimes\mathbb{Q}$, $L=End(A)$ and $G=Aut(A)$
Thanks to A.2.5 there exists a maximal order M inside F. Then we can assume L is a M submodule of F. Indeed if we define L' as the A submodule of F generated by L and G' as the set of automorphisms of F as F module, i.e. the invertible elements of F, such that $\sigma(L')=L'$, we have that L is of finite index in L' and hence G is of finite index in G'. So for the finiteness statement we can replace L with L' and G with G'.
Now by A.2.4 there exists a finite number of right M submodules of F such that their additive group is isomorphic to $\mathbb{Z}^t$ for every t and so there exists a finite number of couple of A submodule of F such that $L_1\oplus L_2\simeq L$. Denote this last set set with W.
Now if N is a submodule of F, define $L_1(N)=N\cap L$ and $L_2(N)=\frac{L}{L_1}$. By A.2.5 we have that $L\simeq L_1(N)\oplus L_2(N)$, so that we have a map $V\to H$ and we have just to show that it is injective. But if $L_1(N)$ is isomorphic to $L_1(N')$ and $L_2(N)$ is isomorphic to $L_2(N')$ then, taking direct sum we get an isomorphism $L\to L$ and so, tensoring with $\mathbb{Q}$, an isomorphism $\sigma: F\to F$ such that $\sigma(L)=L$ and $\sigma(N)=N'$

□

Corollary 3.1.6. *When we prove (2) of 1.2.3 we can assume $d=1$.*

3.2 Theta groups

In this section we introduce the notion of Theta group. It will be helpful to understand how to construct principal polarization over finite extension of the ground field and also, in the next chapter, to construct the moduli space of Abelian varieties. We fix a field k, an abelian variety with a separable polarization L and we start with a definition.

Definition 3.2.1. The theta group associated to L is the functor

$$G(L)(T):=\{(x,\psi)\ |x\in K(L)(T) \text{ and } \psi:\ L\simeq t_x^*L\}$$

The following are direct consequences of the definition.

Remark.
1)$G(L)$ is a group under the law $(x,\psi)(y,\eta):=(x+y,t_y^*\psi\circ\eta)$
2)We have an exact sequence of group functors:

$$0\to\mathbb{G}_m\to G(L)\to K(L)\to 0$$

where the first map send x in $(0,m_x)$, where m_x is the fiber wise multiplication by x, and the second is the natural projection $(x,\psi)\mapsto x$
3)$\mathbb{G}_m$ is central in $G(L)$

If L is a line bundle, we denote with $\mathbb{L}$ the associated geometric line bundle and $\mathbb{L}^*$ the associated $\mathbb{G}_m$ torsor.

Proposition 3.2.2. *$G(L)$ is representable.*

Proof. Since $K(L)$ is representable we have just to show that the morphism $G(L)\to K(L)$ is representable. So fix a scheme $x:T\to K(L)$ an define $M=L_T\otimes t_x^*L_T^{-1}$. Using A.3.4 and the fact that x factor trough $K(L)$ we get that $p_{T,*}M$ is locally free of rank one, so that we can consider $\mathbb{M}^*$. For every T scheme T', the $G(L)$ point of T' that commutes with the map $T\to G(L)$, are exactly the non vanishing section of $M_{T'}$ i.e the map from T' to M^*. □

We are interested in $G(L)$ thanks to the following

Proposition 3.2.3. *Let H be a finite subgroup of A and $f:A\to B:=\frac{A}{Y}$. There is a bijection between morphism of groups $H\to G(L)$ lying over the natural inclusion $H\to A$ and $M\in Pic(B)$ such that $f^*M=L$*

Proof. It is enough to observe that, by definition, a morphism $H\to G(L)$ lying over the natural inclusion is an action of H on L compatible with the natural action of H on A. □

With this we are ready to give a proof of the statement of our interest:

Theorem 3.2.4. *Let K of characteristic zero. Every abelian variety is isogenous to a principally polarized one over a finite extension of k.*

Remark. The characteristic zero assumption it is not needed, but makes the proof more elementary and we will need this theorem only in this situation.

Proof. We can assume $k = \overline{k}$. We take any ample line bundle L on A and a maximal isotropic, for the pairing induced by the polarization, subspace H of L. By 3.1.3, the polarization induced by L descend to a polarization λ of $B = \frac{A}{H}$, represented by some line bundle M, and, by maximality, there aren't subgroup containing H with the same propriety. We claim that the λ is principal. Assume by contradiction that $K(M)$ is not trivial. Then, since we are working over an algebraically closed field of characteristic zero, there is a subgroup of $K(M)$ in the form $\frac{\mathbb{Z}}{l\mathbb{Z}}$ for some prime l. Denoting with T the pullback of $G(M)$ along the inclusion $\frac{\mathbb{Z}}{l\mathbb{Z}} \to K(M)$, we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_m \arrow[r] & G(M) \arrow[r] & K(M) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[u] & T \arrow[r] \arrow[u] & \frac{\mathbb{Z}}{l\mathbb{Z}} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Observe that T is commutative, since $\mathbb{G}_m$ is central commutative and $\frac{\mathbb{Z}}{l\mathbb{Z}}$ is cyclic. Moreover $\mathbb{G}_m = k^*$ is divisible and so the sequence split. So we get a map $\frac{\mathbb{Z}}{l\mathbb{Z}} \to T$ and hence a map $H \to G(M)$ lying over the natural inclusion. But this means that we can lift the polarization to $\frac{B}{\frac{\mathbb{Z}}{l\mathbb{Z}}}$ and this is not possible by construction.

∎

Remark. With the same ideas one can prove that every abelian variety is isogenous, in a finite extension, to a one with a principal polarization given by an symmetric line bundle.

3.3 Rosati involution

Definition 3.3.1. Given a polarized abelian variety (A, λ), we define the Rosati involution

$$\dagger : End(A) \otimes \mathbb{Q} \to End(A) \otimes \mathbb{Q}$$

as $f^\dagger = \lambda^{-1} f' \lambda$.

Proposition 3.3.2. *Fix (A, ψ_L) with L ample. The Rosati symmetric bilinear form of $End(A) \otimes \mathbb{Q}$ associated to ψ_L, that maps (f, g) to $Tr(f^\dagger g)$, is positive definite.*

Proof. Denote $P_{ff^\dagger}(t) = \sum_{0 \leq i \leq 2g} a_i t^i$ the characteristic polynomial of $ff^\dagger$. We have to compute $Tr(ff^\dagger)$ and this is $-a_{2g-1}$. Thanks to 2.0.1 we have that

$$Deg(\psi_{f^*L^{-1} \otimes L^n}) = \chi(f^*L^{-1} \otimes L^n)^2 = \Big(\frac{(nc_1(L) - c_1(f^*L))^g}{g!}\Big)^2 =$$

$$= \Big(\frac{\sum_{0 \leq i \leq g} \binom{g}{i} (-1)^{g-i} (c_1(L)^i c_1(f^*L)^{g-i})}{g!}\Big)^2$$

But we have also

$$Deg(\psi_{f^*L^{-1} \otimes L^n}) = Deg(n\psi_L - \psi_{f^*L}) = Deg(\psi_L n - f'\psi_L f) =$$

$$= Deg(\psi_L n - \psi_L f^\dagger f) = Deg(\psi_L) Deg(n - f^\dagger f)) = \chi(L)^2 P_{f^\dagger f}(n)$$

. Comparing the coefficient of the two polynomials we get

$$Tr(ff^\dagger) = \frac{2\chi(L)^{-2} g c_1(L)^g (c_1(L)^{g-1} c_1(f^*L))}{(g!)^2} = \frac{2g(c_1(L)^{g-1} c_1(f^*L))}{c_1(L)^g}$$

and this is positive since L is ample and f^*L is effective (f is a flat map on the image). ∎

Proposition 3.3.3.
1)$Aut((A,\lambda))$ is finite
2)Every element of $Aut((A,\lambda))$ that acts as the identity over the $n>2$ torsion points is the identity.

Proof. 1) Observe that if $\alpha \in Aut((A,\lambda))$ then $\alpha^\dagger\alpha = 1$ so that

$$\alpha \in End(A) \cap \{\beta \in End(A)\otimes\mathbb{R} \mid Tr(\alpha^\dagger\alpha) = 2g\}$$

. The first is discrete and the second is compact thanks to 3.3.2 and so we are done.
2)Let α as in the statement. Then by the previous point it is of finite order, so that its eigenvalues are roots of unit. Moreover $\alpha - 1 = n\beta$ and α is unipotent thanks to the following

Claim: if α is a root of unit, β an algebraic integer and $\alpha - 1 = n\beta$ with $n > 2$ then $\alpha = 1$.

Proof. If this is not true we can assume that α is a p root of unit with p prime (α^m is a p-root of 1 different from 1 for some m and then we get $\alpha^m - 1 = (\alpha-1)c = n\beta c$ for some algebraic integer c). We have

$$p = N_{\mathbb{Q}(\alpha)|\mathbb{Q}}(\alpha - 1) = N_{\mathbb{Q}(\alpha)|\mathbb{Q}}(n\beta) = n^{p-1}N_{\mathbb{Q}(\alpha)|\mathbb{Q}}(\beta)$$

and this is not possible if $n > 2$. □

So α is unipotent and β is nilpotent. Now $\beta_1 = \beta^\dagger\beta \neq 0$ thanks to 3.3.2. Moreover $\beta_1 = \beta_1^\dagger$ so that, thanks to 3.3.2, $\beta_1^2 \neq 0$. Similarly $\beta_1^{2m} \neq 0$ for every m and this is not possible since β is nilpotent. □

All this lemmas are useful for the following corollary. Recall that, in the setting of 1.2.3, the all the B_n have a polarization of degree d.

Corollary 3.3.4. *Suppose that we have a family B_n of abelian variety isogenous to each other and with a polarization λ_n of degree d. If they fall into finitely many isomorphism classes as polarized abelian variety over $\overline{K}$ then they fall into finitely many classes as abelian varieties over K.*

Proof. Suppose that we know that there exist finitely many isomorphism classes as polarized abelian variety in the algebraically closure. Then we choose some $m > 2$ coprime with the characteristic of the field and a finite extension F of K such that all the B_n have the m torsion points rational. We claim that all of them are isomorphic over F. Indeed let α be an isomorphism over the algebraic closure of K. Then, for every $\sigma \in Gal(\overline{K}|F)$, $\sigma(\alpha)\circ\alpha^{-1}$ is the identity over the point of order m, since they are F rational, and hence it it is the identity, by 3.3.3. So $\sigma\alpha = \alpha$ for all $\sigma \in Gal(\overline{K}|F)$ and hence α is defined over F. Now we can easily conclude showing that the map

$$\{\text{ polarized abelian variety over K up to iso}\} \to \{\text{ polarized abelian variety over F up to iso }\}$$

has finite fiber. In fact the fiber of this map over an element (B,λ) is parametrized, thanks to A.1.17, by $H^1(Gal(F|K), Aut((B,\lambda))$ that is finite, thanks to 3.3.3. □

We will need another couple of properties of the Rosati involution. Denote with $NS(A) = \frac{Pic(A)}{Pic^0(A)}$ the Neron-Severi group of A and observe that we have a canonical embedding $NS(A)\otimes\mathbb{Q} \to Hom(A,A')\otimes\mathbb{Q}$ that send M to ϕ_M. Moreover, if ψ_L is a polarization of A, we have an isomorphism $Hom(A,A')\otimes\mathbb{Q} \simeq End(A)\otimes\mathbb{Q}$ that send f to $\phi_L^{-1}\circ f$. So we get an injective map $NS(A)\otimes\mathbb{Q} \to End(A)\otimes\mathbb{Q}$ that send M to $\psi_L^{-1}\circ\psi_M$

Proposition 3.3.5. *Let α be the image of some ample line bundle M. Then:*
1)α is fixed by the Rosati involution induced by L.
2)$\mathbb{Q}(\alpha)$ is a direct sum of totally real field.
3)If α is symmetric all the component of α in the previous decomposition are positive.

Proof. 1)This is clear since $\psi_L^{-1}(\psi_L^{-1}\circ\psi_M)'\psi_L = \psi_L^{-1}\psi_M$ thanks to the fact that $\psi_L' = \psi_L$.
2)$\mathbb{Q}(\alpha)$ is a direct sum of fields, since it is a commutative finite dimensional algebra over $\mathbb{Q}$. To show that it is totally real observe that $\mathbb{Q}(\alpha)\otimes\mathbb{R} \simeq \mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$ and that every element in $\mathbb{Q}(\alpha)$ is fixed by the Rosati involution. By continuity, the trace bilinear form is positive semidefinite in $\mathbb{Q}(\alpha)\otimes\mathbb{R}$, and hence positive definite since the quadric is defined over $\mathbb{Q}$ and hence the null space has to define over $\mathbb{Q}$. So we get that if $x \in \mathbb{Q}(\alpha)\otimes\mathbb{R}$, $Tr(x^2) = Tr(xx^\dagger) > 0$. But this implies that there are no complex embedding, since if $i^2 = -1$, $Tr(i^2) = -Tr(1) < 0$.
3)Write the characteristic polynomial of $\alpha = \psi_L^{-1}\circ\psi_M$ as $P_\alpha = \sum_{0\le i\le 2g}(-1)^i a_i t^i$. We will show that

$a_i > 0$. Since all of its roots are real this is enough, thanks to the Descartes' sign rule. The reasoning is similar to the one done in 3.3.2.
Indeed for every n we get:

$$\chi(L^n \otimes M^{-1})^2 = deg(\psi_{L^n \otimes M^{-1}}) = deg(n\psi_L - \psi_M) =$$

$$= deg(\psi_L) deg(n - \psi_L^{-1}\psi_M) = deg(\psi_L) P_\alpha(n)$$

Moreover Riemann Roch shows that

$$\chi(L^n \otimes M^{-1}) = \frac{1}{g!}((c_1(L^n \otimes M^{-1})^g) = \sum_{0 \le i \le g} (-1)^v \frac{c_1(L)^{g-i} c_1(M)^i}{(g-i)!i!} n^{g-i}$$

Comparing the coefficient gives the assertion. $\square$

Chapter 4

Proof over function fields

The proof over function field is made in two steps. First of all we will prove the conjecture over function fields of degree of transcendence 1. Here the proof use the notions of height to prove the finiteness statement. Indeed, we will construct a subspace of some $\mathbb{P}^n$ that parametrize abelian variety with a fixed polarization. Then we will show that the height of the points in this projective space that correspond to a family of abelian variety isogenous to each other is bounded, and hence we will get the finiteness from A.3.2. To extend the result over arbitrary function field we will need some induction argument on the transcendence degree of the function field.

4.1 Mumford moduli space

We give a sketch of the construction of the moduli space of Abelian variety over a field. We will work with the moduli space of polarized abelian variety with a delta structure. We don't need the whole construction of the moduli space, since we have just to deal with the k point of it. Also with these restrictions the proof involves some long and hard computations on line bundles. We will present the main ideas of the construction and some computations. For the complete proof we refer the reader to [Mum66]. From now on we will assume that k is an algebraically closed field of characteristic different from 2 and we will deal with abelian variety with a fixed polarization of some degree d coprime with the characteristic of the field.
The basic idea of the construction of Mumford is to construct, fixed an abelian variety A with very ample line bundle L, a "canonical" basis of $H^0(A, L)$, so that we can embed A in a canonical way in $\mathbb{P}^n$. Then one shows that the equation defining A inside the projective space are uniquely determined by the image of the neutral element of A in $\mathbb{P}^n$. So, after fixing some extra structure, we can recover A as polarized abelian variety from a unique point of $\mathbb{P}^n$.
So fix an abelian variety A with a separable ample line bundle L. The perfect paring induced by L give us a decomposition of $K(L)$ as direct sum of two maximal isotropic orthogonal subgroup $K_1(L), K_2(L)$, where $K_1(L)$ is a maximal isotropic subspace. If $d = (d_1, ..., d_n)$, with $d_i|d_{i+1} \in \mathbb{Z}$, we denote with $K(d) = \oplus_{1\leq i\leq n} \frac{\mathbb{Z}}{d_i\mathbb{Z}}$ and $H(d) = K(d) \oplus K(d)'$. Recall that we have an exact sequence of groups:

$$0 \to k^* \to G(L) \to K(L) \to 0$$

To put it in a canonical form, we define $G(d) := k^* \times K(d) \times K(d)'$ with the group structure given by $(x, y, z)(x', y', z') = (xx'z'(y), y + y', z + z')$. We have an exact sequence

$$0 \to k^* \to G(d) \to H(d) \to 0$$

Definition 4.1.1. L is said to be of type $d = (d_1, ..., d_n)$, if there exists an isomorphism $K_1(L) \simeq K(d)$. A delta structure on (A, L) is the choice of an isomorphism between the two exact sequences that is the identity over k^*.
A level subgroup of $G(L)$ is a subgroup H such that $H \cap k^*$ is zero.

Lemma 4.1.2. *There is a bijection between level subgroups of $G(L)$ and couples (f, α), where $g : X \to Y$ is an isogeny and α is an isomorphism $f^*M \simeq L$ where $M \in Pic(Y)$*

Proof. This follows directly from 3.2.3. □

Lemma 4.1.3. *The set of delta structure is non empty and finite.*

Proof. It is clear that it is finite and that every line bundle has a type d, just choose a maximal isotropic subgroup $K_1(L)$ of $K(L)$. Choose an isomorphism $K_1(L) \simeq K(d)$ and observe that it induces an isomorphism $K_2(L) \simeq K(d)'$. If we choose level subgroups over $K_1(L)$ and over $K_2(L)$ (observe that they exists thanks to 4.1.2 and 3.2.3), we can define a map $G(d) \to G(L)$ that it is the identity over k^* and that gives us the desired isomorphism by the snake lemma. □

The key idea of Mumford, that will allow us to chose a canonical basis of $H^0(A, L)$, is the following.

Proposition 4.1.4.
- *There is an unique irreducible representation $V(d)$ of $G(d)$ such that k^* acts as its natural character.*
- *Every other representation of $G(d)$ in which k^* acts in this way is isomorphic to $V(d)^r$, where $r = dim(V^{\tilde{K}(d)})$.*
- *$H^0(A, L)$ is an irreducible representation of $G(d)$*

Proof.
- Suppose that V is an irreducible representation and consider $K(d)$ as a subgroup of $G(d)$. Since $K(d)$ is commutative and with cardinality not divisible by characteristic of the field we have that
$$V = \oplus_{\chi \in Hom(K(d),k^*)} V_\chi$$
as K representation, where V_χ is the subspace made by $v \in V$ such that $\chi(k)v = kv$ for every $k \in K(d)$.
First of all observe that $\frac{G(d)}{k^* K(d)} \simeq Hom(K(d), k^*)$. Indeed a simple computations show that, for every $y \in G(d)$, there exists a well defined character $\chi_y : K(d) \to k^*$ such that $\chi_y(k)k = y^{-1}ky$ for every $k \in K(d)$. Since k^* is central and K is commutative, the homomorphism $G(d) \to Hom(K, k^*)$ that send y to χ_y gives us an injective map $\eta : \frac{G(d)}{k^* K} \to Hom(K, k^*)$ that is an isomorphism for cardinality reasons.
Moreover one easily check that if $v \in V_\chi$ then yv is in $V_{\chi * \chi_y}$ for every $y \in G(d)$. So we get that all the V_χ are different from zero, since at least one is different from 0. If we fix a $v \in V_0$, we get that the subspace generated by yv, while y is varying, intersected with V_χ is one dimensional. Since the representation is irreducible, all the V_χ are one dimensional.
We are ready to conclude. Indeed consider the subgroup $H = k^* \times K(d)'$ and the representation W which is k with the natural action of k^* and the trivial action of $K(d)'$. Then we have a canonical morphism $Ind_G^H(W) \to V$, by the universal property of Ind_G^H and the fact that k^* acts as its natural character over V. Since $V_0 \neq 0$, the morphism is non trivial. Since V is irreducible the morphism is surjective and so we conclude counting dimension.
- Observe that the element such that $x^m = 1$ form a finite subgroup of $G(m)$ of $G(d)$, where $m = |K(d)|$. Since $k^*G(m) = G(d)$ a $G(d)$ representation is semisimple if and only it is semisimple as $G(m)$ representation. Since char(k) does not divide m and $G(m)$ is finite, all the $G(m)$ representation are semisimple and so we are done. For the second statement it is enough to observe that $Dim(V(d)^K) = Dim(V_0) = 1$ by the proof of the previous point.
- We have a natural action of $G(L)$ given by $(x, \psi)l = t_{-x}\psi(l)$ in which k^* acts as it's natural character. By the previous two points it is an irreducible representation of $G(L)$.

□

So to construct a "canonical" basis of $H^0(A, L)$ it is enough to construct a canonical representation $V(d)$ of $G(d)$ in which k^* acts as the natural character with a canonical basis. Then by Schur's lemma, there is a unique, up to scalar multiplication, equivariant isomorphism $V(d) \simeq H^0(A, L)$ and hence a unique isomorphism $\mathbb{P}(H^0(A, L)) \simeq \mathbb{P}(V(d))$. To construct such a representation define
$$V(d) := \{K(d) \to k\}$$
, as the set of function from $K(d)$ to k, and an action of $G(d)$ on it as
$$(x, y, z)f(w) = xz(w)f(y + w)$$
Observe that we have a canonical basis of $V(d)$ made by the function e_a defined by, for every $a, b \in K(d)$, $e_a(b) = \delta_{a,b}$. By counting dimension this is irreducible by the previous theorem. So we have proven that

if we have (A, L) and a delta structure we can construct a canonical map $A \to \mathbb{P}^n$ where $n = deg(L)$, thanks to 2.0.1.

Definition 4.1.5. Given (A, L) with a delta structure we define the delta null coordinates $\{q_L(a)\}_{a \in K(d)}$ as the coordinates of the zero element of A in the previous embedding.

From now on suppose also that L is symmetric (this is not a big complication since we can change L with $L \otimes (-1)^* L$ and observe that this is still separable since $K(L \otimes i^* L) = K(L^2)$ (use that one can write every line bundle as a product of a symmetric one and one in Pic^0, see [MVdG13] Ex 7.3)). Mumford proved the following:

Theorem 4.1.6. *Suppose that $A[4] \subseteq K(L)$. Then L is very ample and the equation defining A inside $\mathbb{P}^n$ are quadratic polynomials with coefficients that depends only on $\{q_L(a)\}_{a \in K(d)}$. In particular the isomorphism class of the polarized abelian variety depends only on the coordinates of the image in $\mathbb{P}^n$ of the neutral element of A.*

Proof. This is the main result of [Mum66]. □

Remark. Observe that also the restriction on the torsion is not a big problem. Indeed if we change L with $M = (L \otimes (-1)^* L)^2$ this will always be totally symmetric and $A[4]$ will be included in $K(M)$.

During the proof of the theorem Mumford proved also some useful relations that we will need in the sequel.

Proposition 4.1.7. *Let N be a subgroup of $K(d)$ and suppose there is an isogeny $f : A \to \frac{A}{N}$ and a line bundle M such that $f^* M = L$. Then there exists a delta structure on $(\frac{A}{N}, M)$ such that the delta null values associated coincide with $\{q_L(a)\}_{a \in N^\perp}$.*

Proof. See [Mum66] and [Zar73b] □

Remark. Observe that the type of L^2 is made by even numbers thanks to the fact that $K(L) = 2K(L^2)$ and the Riemann Roch theorem.

Proposition 4.1.8. *If (A, L^2) is a symmetric separable polarized abelian variety with delta structure $K(2d)$ and Mumford coordinates $(q_{L^2}(a))_{a \in K(2d)}$. Then there exists a delta structure $K(d)$ on (A, L) such that $K(d) \subseteq K(2d)$ in the natural way. Denote with $K(2)$ the 2 torsion of $K(2d)$ and of $K(d)$. Then we have the following formulas:*
1)$q_L(x + \eta) = 2^{-g} \sum_{l \in K(2)'} l(\eta) \theta_L(x, l)$ where $\theta_L(x, l) = \sum_{\omega \in K(2)} l(\omega) q_{L^2}(x + \omega)$ and η in $K(2)$.
2)$q_L(u + v) q_L(u - v) = \sum_{\eta \in K(2)} q_{L^2}(u + \eta) q_{L^2}(v + \eta)$ if $u, v \in K(2d)$ with $u + v \in K(d)$.
3)$\theta_{L^2}(x, l)^2 = \sum_{\eta \in K(2)} l(\eta) q_L(2x + \eta) q_l(\eta)$
4)$q_L(u)^2 = \sum_{\eta \in K(2)} q_{L^2}(u + \eta) q_{L^2}(\eta)$

Proof. See [Mum66] Chapter 3 and [ZM72] last page. □

Remark. Suppose that K is not algebraically closed. Passing on the algebraic closure we again find a delta structure and hence some coordinates. Observe that this coordinates are defined over a finite extension of K such that the group scheme $K(L)$ becomes constant.

4.2 The case of curves

The proof of Zarhin relies on the study of heights of families of isogenous abelian varieties. Suppose that A is an Abelian variety over the function field of projective smooth curve over $\mathbb{F}_q$. This is the case every time K is a finitely generated field over $\mathbb{F}_q$ with transcendence degree 1. Since it is a global field, we have a notion of height in the projective space, see *A*.3.1, and hence, thanks to the previous section, of height of a polarized abelian variety with a delta structure. Since it is of characteristic p every valuation is non Archimedean. These are the two key points in the proof. As we will seen in a moment, the non Archimedean property of the valuations will allow us to show that the height does not change under isogeny and so to deduce the finiteness theorem from the Northcott's propriety of the height, A.3.2.
If A is an abelian variety with a polarization L of degree d we define the delta height of A, $d(A, L)$, as the height of the delta null point of $(A, (L \otimes i^* L)^2)$ where we choose some delta structures on $M = (L \otimes i^* L)^2$ and on M^2 as in 4.1.8. We will change always change L with M, so that we will assume L symmetric and $A[4] \subseteq K(L)$. Moreover if we have some separable isogeny $f : A \to B$ and a line bundle such that $f^* N = L$, we choose some delta structure on (B, M) as in 4.1.7.

Remark. Different choices of the delta structure does not change the height. Indeed a different isomorphism between $K(L)$ to $K(d)$ can always be performed at the level of the some finite extension of the finite base field. Since all the valuation are trivial on this field this isomorphism does not affect the height.

We begin with the key lemma:

Lemma 4.2.1.
1)$d(A,L)=d(A,L^2)$
2)$d(A,L)=h(q_L(a))_{a\in K(2)}$.

Proof. 1) For every valuation v We have

$$max_{u\in K(d)}(|q_L(u)^2|_v)=max_{u\in K(d)}(|\sum_{\eta\in K(2)}q_{L^2}(u+\eta)q_{L^2}(\eta)|_v)\leq$$

$$\leq max_{u\in K(d)}(max_{\eta\in K(2)}(|q_{L^2}(u+\eta)q_{L^2}(\eta)|_v))\leq max_{u\in K(2d)}|q_{L^2}(u)^2|_v$$

thanks to 4.1.8 and the fact that all the valuations are not Archimedean. As a consequence $2d(A,L)\leq 2d(A,L^2)$. For the other inequality we have

$$max_{u\in K(2d)}(|q_{L^2}(u)^2|_v)\leq max_{x\in K(2d),l\in K(2)'}(|\theta_{L^2}(x,l)^2|_v)=$$

$$=max_{x\in K(2d),l\in K(2)'}(|\sum_{\eta\in K(2)}l(\eta)q_L(2x+\eta)q_l(\eta)|_v)\leq max_{x\in K(d)}|q_L(x)^2|_v$$

as before, noticing that $l(\eta)$ has norm one since it is a roots of unit and that the first inequality follows from the first point of 4.1.8. As a consequence $2d(A,L^2)\leq 2d(A,L)$ and we are done.
2)

$$(max_{u\in K(d)}(|q_L(u)|_v)^2=max_{u\in K(d)}(|q_L(u)|_v)\leq max_{u\in K(d),\eta\in K(2)}(|q_{L^2}(u+\eta)|_v)max_{u\in K(d)}(|q_{L^2}()|_v)\leq$$

$$\leq max_{u\in K(d),\eta\in K(2)}(|q_{L^2}(u+\eta)|_v)max_{u\in K(2d)}(|q_{L^2}(u)|_v)$$

But, thanks to the previous point, this implies that $max_{u\in K(d)}(|q_L(u)|_v)=max_{u\in K(2d)}(|q_{L^2}(u)|_v)\leq max_{u\in K(2)}(|q_{L^2}(u)|_v)$. ☐

From now on the proof will be formal.

Proposition 4.2.2. *1)If $f:A\to B$ is a separable isogeny then $d(A,f^*L)\geq d(B,L)$*
2) If $f:A\to B$ is a separable isogeny then $d(A,f^*L)=d(B,L)$
3) If L_1 and L_2 are two separable line bundles over A then $d(A,L_1)=d(A,L_2)$

Proof. Let N be the kernel of f.
1)Using 4.1.7 we get that set of coordinates of B in which we can compute the height of B is a subset of the set of coordinates of A and so we are done.
2)Up to base change, that does not alter the height, every isogeny can be factorized as product of an isogeny of degree 2^n and an isogeny of odd degree.
If the degree of the isogeny is odd then, $K(2)\subseteq N^{\perp}$ and hence we are done thanks to 4.1.7 and the second point of the previous lemma. If the degree of the isogeny is 2^n the we have a map $g:B\to A$ such that $f\circ g=2^n$. Then we get that, thanks to the first point of the proposition, that $d(B,L^{2^{n^2}})=d(B,(f\circ g)^*(L)\geq d(A,f^*L)\geq d(B,L)$ hence we conclude thanks to the previous lemma that tell us that $d(B,L^{2n^2})=d(B,L)$.
3)We use again a sort of Zarhin trick to reduce this statement to the previous point. Consider the embedding $NS(X)\otimes\mathbb{Q}\to End(X)\otimes\mathbb{Q}$ as in 3.3.5 given by L_2. L_2 is sent to 1 and L_1 is sent to some α totally real and totally positive element in $End(X)^0$. Using the previous point we can replace L_1 with $L_1^{n^2}$ for some n coprime with the characteristic of the field and hence we can assume that $\alpha\in End(A)$. Now, in $\mathbb{Q}[\alpha]$, the equation $X_1^2+X_2^2+X_3^2+X_4^2=\alpha X_5^2$ has a solution, since in has a solution in every completion and it satisfies, as every quadratic, the Hasse principle (for the finite places is clear, for the infinite place we have to use that the component of α are all positive). In particular the quadric $X_1^2+X_2^2+X_3^2+X_4^2=4\alpha$ is rational and so it satisfies weak approximation. Observe that, since $p\neq 2$, it as also a $\mathbb{Z}_p$ point namely $(\alpha+1,u(\alpha-1),v(\alpha-1),0)$, where $u,v\in\mathbb{Z}_p$ are such that $u^2+v^2=-1$. So in $\mathbb{Q}_p[\alpha]$ we have a solution such that p does not divide the denominator and hence, thanks to weak

approximation, we get a solution in $\mathbb{Q}[\alpha]$ with the same property. Multiplying this solution for a big N coprime with p we can find $a, b, c, d \in Z[\alpha]$ such that $a^2 + b^2 + c^2 + d^2 = N^2 4\alpha$.
Consider now the endomorphism ψ of X^4 given by the matrix:

$$M = \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix}$$

We have

$$M^t M = N^2 4\alpha Id = (2N)^2 \alpha Id$$

and hence $\psi^*(\otimes_{1 \le i \le 4} \pi_i^* L_2) = (\otimes_{1 \le i \le 4} \pi_i^* L_1)^{(2N)^2} = (2N)^*(\otimes_{1 \le i \le 4} \pi_i^* L_1)$ (see [MVdG13] ex. 7.8), up to some element in $Pic^0(A)$ that does not alter the delta structure, and so we get what we want thanks to the previous point and the additivity of the height with respect to tensor product of line bundle. $\square$

With this in our hand we can prove the finiteness statement

Theorem 4.2.3. *(Zarhin) If K is a function field of transcendence degree equal to 1 over a finite field of characteristic $p \neq 2$, then the Tate conjecture is true for every abelian variety and every l different from p.*

Proof. We will use the second point of 1.2.3. Thanks to 3.1.4 we can assume that A has a symmetric polarization of degree d coprime with the characteristic of the field.
We choose an extension F such that a delta structure is defined for every abelian variety isogenous to A with a polarization of degree d and the d torsion rational, using A.1.16. Now by the previous proposition all of them have the same height and so they fall into finitely many classes in the algebraic closure. So by 3.3.4 they fall in to finitely many isomorphism class over F. Now just observe that, as in the proof of 3.3.4 using A.1.17 and 3.3.3, this implies that they fall into finitely many classes over K. $\square$

4.3 Reduction to the case of curves

4.3.1 Fullness

We will now extend the result to every function field with an induction argument. We will need the following lemma of commutative algebra.

Lemma 4.3.1. *Let R be a Dedekind domain, $n \in \mathbb{N}$, I any set, N a finitely generated R module, $\{f_i : M_i \to N\}$ a family of maps of R modules such that, for every i, $coker(f_i)$ is flat and F a flat module. Then*

$$(\cap_{i \in I} Im(f_i)) \otimes F = \cap_{i \in I}(Im(f_i) \otimes F)$$

Proof. $\square$

We have the following commutative diagram with all the map injective, thanks to the flatness of F

$$\begin{tikzcd} \cap_{i \in I} Im(f_i)) \otimes F \arrow[rr] \arrow[dr] & & \cap_{i \in I}(Im(f_i) \otimes F) \arrow[dl] \\ & N \otimes F & \end{tikzcd}$$

So we have just to show that $\cap_{i \in I}(Im(f_i) \otimes F) \subseteq (\cap_{i \in I} Im(f_i)) \otimes F$. To prove this is enough to prove that there exists a finite subset $J \subseteq I$ such that $\cap_{i \in I} Im(f_i) = \cap_{i \in J} Im(f_i)$. Indeed given this we get that

$$\cap_{i \in I}(Im(f_i) \otimes F) \subseteq \cap_{i \in J}(Im(f_i) \otimes F) = \cap_{i \in J}(Im(f_i)) \otimes F = \cap_{i \in I}(Im(f_i)) \otimes F$$

To prove the existence of such a J, we will show that the set of subset in the form $\cap_{i \in J} Im(f_i)$ with J finite, satisfies the descending chain condition, this is enough thanks to Zorn Lemma. So consider a descending chain $\cap_{i \in J_k} Im(f_i)$ with $J_k \subseteq J$ finite. Taking quotient, we get a sequence of surjective maps

$$\ldots \to \frac{N}{\cap_{i \in J_k} Im(f_i)} \to \frac{N}{\cap_{i \in J_{k-1}} Im(f_i)} \to \ldots$$

To conclude we observe that, since we are over a Dedekind domain and, by induction every $\frac{N}{\cap_{i\in J_k} Im(f_i)}$ is torsion free and hence projective, this sequence stabilize (a surjective map between two projective modules of the same rank is an isomorphism). $\square$

Remark. Suppose that $(R, \mathfrak{m})$ is a commutative henselian noetherian domain of dimension bigger then 2 and A a finite unramified cover. Suppose that there is a map $f : A \to k(\mathfrak{m})$ that factorize trough all the ideals of height 1 of R. Then, since $\cap_{ht(\mathfrak{p})=1}\mathfrak{p} = 0$, $\frac{A}{\mathfrak{p}}$ and $k(\mathfrak{m})$ are connected, f factorize trough R.

Now we can state the theorem

Theorem 4.3.2. *Let R a normal noetherian domain of dimension bigger or equal to 2 such that:*
1)the set $C(R) := \{char(\frac{R}{\mathfrak{p}}\ \mathfrak{p} \in Spec(R)\} - \{0\}$ is finite
2)for each prime $\mathfrak{p}$ of height 1 the T_l is faithful for $Frac(\frac{R}{\mathfrak{p}})$.
3) the closed point with height bigger then 2 are dense in $Spec(R)$
Then for every abelian variety A over R, the Tate conjecture is true for $A_{Frac(R)}$ over $Frac(R)$

Corollary 4.3.3. *T_l is faithful for every finitely generated field K of characteristic $p > 2$.*

Proof. Let A be an abelian variety over K.
Suppose that $char(K) = p > 0$ and the finite base field is k. Then we have what we want by induction on the transcendence degree of the field K over k and the Tate conjecture for function field of transcendence degree 1 over k, observing that if R is a normal model of k over a finite field such that the abelian variety A extend to an abelian variety over R, then for every $\mathfrak{p} \in Spec(R)$ of height 1, $Tr.degree_k(Frac(\frac{R}{\mathfrak{p}})) = dim((\frac{R}{\mathfrak{p}}) \leq dim(R) - 1 = Tr.degree_k(Frac(R)) - 1$. Observe that the third hypothesis is clearly satisfies since the closed point are dense and all of the maximal height. $\square$

Now we prove the theorem. We denote with $(H, \mathfrak{m}_H)$ the strict henselianization of $(R)_y$ where y is a closed point of maximal height of $Spec(R)$. We define the following set:

$$M = \left\{ \begin{array}{c} (S, \mathfrak{m}_S) \text{ where } H \subseteq S \subseteq \overline{K}, \text{ H is a strictly henselian ring and} \\ \mathfrak{m}_S, \text{ the maximal ideal of S, is such that } \mathfrak{p}_S := \mathfrak{m}_S \cap H \text{ has height one.} \end{array} \right\}$$

Denoting with k_S the residue field of S, we define also the obvious maps, for every S, as in the following commutative diagram:

$$\begin{tikzcd}
k_H & \frac{H}{\mathfrak{p}_S} \arrow[l, "\pi_{\frac{H}{\mathfrak{p}_S}}"'] \arrow[d, "r_S"] & H \arrow[l, "\pi^S_H"'] \arrow[ll, bend right, "\pi_H"'] \arrow[r, "i_H"] \arrow[d, "j_S"] & \overline{K} \\
 & k_S & S \arrow[l, "\pi_S"'] \arrow[ur, "i_S"'] &
\end{tikzcd}$$

Moreover we define $G_S = \{g \in G \text{ such that } g(S) = S\}$ and $\mathfrak{q}_S = R \cap \mathfrak{m}_S$. Finally we define for every map $f : X \to Y$ of R algebras, the map $EA_l(f) : End(A_X) \to End(A_Y)$. Observe that the Tate module of all this scheme are isomorphic and in the sequel, to simplify the notation we will identify all of them and they endomorphism algebra. For a proof without any identification see B.0.1.

We start observing that, to prove the Tate conjecture, it is enough to prove that $End_{\Gamma_K}(T_l(A_{\overline{K}}) \subseteq End(A_{\overline{K}}) \otimes \mathbb{Z}_l$, since if we know this we get $End_{\Gamma_K}(T_l(A_{\overline{K}})) = End(A_{\overline{K}}) \otimes \mathbb{Z}_l \cap End_{\Gamma_K}(T_l(A_{\overline{K}}))$ but $End(A_{\overline{K}}) \otimes \mathbb{Z}_l \cap End_{\Gamma_K}(T_l(A_{\overline{K}})) = End(A_K) \otimes \mathbb{Z}_l$, observing that only morphism that are fixed by the Galois are the one defined over K.

We start with an element $x \in End_{\Gamma_K}(T_l(A_{\overline{K}}))$. Since the action of G_S on $T_l(\overline{k})$ is compatible with the map π_S and i_S, we get that $x \in End_{G_S}(T_l(A_{\overline{K}}))$ for every S. Now we observe that the natural map $G_S \to \Gamma_{Frac(\frac{R}{\mathfrak{q}_S})}$ is surjective, so that $x \in End_{\Gamma_{Frac(\frac{R}{\mathfrak{q}_S})}}(T_l(A_{\overline{K}})$ for every S. By hypothesis we have that there exists a $f \in End(A_{Frac(\frac{R}{\mathfrak{q}_S})}) \otimes \mathbb{Z}_l$ such that $T_l(f) = x$. Now we will show that this f is in the image of $EA(\pi_H) \otimes \mathbb{Z}_l$. We have:

Lemma 4.3.4. *Suppose that $H \to H'$ is an injective map between two strictly henselian R_y domains. Then map $End(A_H) \to End(A'_H)$ is bijective.*

Proof. Since the two Tate module are isomorphic and the map $End(A_H) \to T_l(A_H)$ is injective, we get that the map $End(A_H) \to End(A'_H)$ is injective. If $End(A_H|H)$ is the finite unramified scheme that parametrize the endomorphism of A_H we get the following commutative diagram:

$$\begin{array}{ccc} End(A_H|H)_{H'} & \longrightarrow & End(A_H|H) \\ \downarrow & & \downarrow \\ Spec(H') & \longrightarrow & Spec(H) \end{array}$$

An element in $End(A_{H'})$ is a map $Spec(H') \to End(A_H)$ and hence, since $End(A_H|H)$ is a finite disjoint union of scheme of the form $Spec(\frac{H}{I}$ for some I ideal of H, and H' is connected, it is a map $\frac{H}{I} \to H'$ for some ideal I of H. But since the map $H \to H'$ is injective we get that $I = 0$ and hence it comes from a section $Spec(H) \to End(A_H|H)$. $\square$

Thanks to this we get that $f \in \cap_{S\in M}(Im(EA(\pi_{\frac{H}{\mathfrak{p}_S}})) \otimes \mathbb{Z}_l)$. The key lemma is the following:

Lemma 4.3.5.
1)We can choose a closed point $y \in Spec(R)$ of height bigger then 2 such that

$$\cap_{S\in M}(Im(EA(\pi_{\frac{H}{\mathfrak{p}_S}})) \otimes \mathbb{Z}_l) = (\cap_{S\in M} Im(EA(\pi_{\frac{H}{\mathfrak{p}_S}}))) \otimes \mathbb{Z}_l$$

2)$Im(EA(\pi_H)) = \cap_{S\in M} Im(EA(\pi_{\frac{H}{\mathfrak{p}_S}}))$

Proof. 1)Using 4.3.1 we just need to find a y such that $coker(Im(EA(\pi_{\frac{H}{\mathfrak{p}_S}})))$ is torsion free for every s. Suppose $mg = EA((\pi_{\frac{H}{\mathfrak{p}_S}}))(f)$, for some $g \in End(A_{k_H}), f \in End(A_{\frac{H}{\mathfrak{p}_S}})$ and $m \in \mathbb{N}$. We can assume m prime and we divide two cases: $m \in C(R)$ or $m \notin C(R)$.
If $m \notin C(R)$ then $A_{\frac{H}{\mathfrak{p}_S}}[m]$ and $A_{k_H}[m]$ are étale , so that, since f is zero in $A_{k_H}[m]$ it is zero in $A_{\frac{H}{\mathfrak{p}_S}}[m]$, since taking special fiber is an equivalence of categories. So we get that $f \in mEnd(A_{\frac{H}{\mathfrak{p}_S}})$ and we are done. If $m \in C(R)$ the situation is more complicated, because we don't know if $A_{\frac{H}{\mathfrak{p}_S}}[m]$ is étale . The key to avoid the problem is the following:

Claim: For every prime p, there exists an open dense subset V of $Spec(R)$, an integer n and a finite subscheme E of $End(X_V[p]|V)$ such the morphism $End(X_V[p^n]|V) \to End(X_V[p]|V)$ factors though E and the diagonal morphism $E \to E \times_V E$ induce an homeomorphism of E into an open and closed subscheme of $E \times_V E$.

Proof. It is enough to prove that over the generic fiber there exists a finite group scheme E and a natural number n such that the morphism

$$End(A_K[p^n]|K) \to End(A_K[p]|K)$$

factors trough a finite group scheme over K. Indeed if we have this we can find an open subset with all the proprieties required except for the condition on the diagonal. But now observe that the diagonal induces an homeomorphism into an open an closed subset on the generic fiber, since it is true when we take the reduced part (they are just disjoint union of point). For this, since the image of the morphism and finiteness condition are stable by base change, we can assume K algebraically closed. Then it is enough to show that $End(A[p^n]) \to End(A_K[p])$ is finite for some n. Now observe that for n big enough, $Im(\varprojlim_i End(A[p^i]) \to End(A[p])) = Im(End(A[p^n]) \to End(A[p]))$. Indeed, for sure we have that $Im(End(A[p^n]) \to End(A[p])) \subseteq Im(End(A[p^n]) \to End(A[p]))$ and the set of image satisfies the descending chain condition, since they are the K points of closed subscheme of a noetherian scheme. But now observe that $\varprojlim_i End(A[p^i]) = End(A[p^\infty])$ and this is finitely generated. $\square$

With this in our end we choose a closed point $y \in \cap_{m\in C(R)} V_m$ of height bigger then 2, where V_m is the open subset of the claim relative to m. We have to show that f is zero on $A_{\frac{H}{\mathfrak{p}_S}}[m]$ knowing that it is zero on $A_{k_H}[m]$. Then we have the following commutative diagram, where Δ is the diagonal:

$$\begin{tikzcd} E_{\frac{H}{\mathfrak{p}_S}} \times E_{\frac{H}{\mathfrak{p}_S}} & E_{k_H} \times E_{k_H} \arrow[l] \\ \Delta_{E_{\frac{H}{\mathfrak{p}_S}}} \arrow[u] \arrow[d] & \Delta_{E_{k_H}} \arrow[l] \arrow[u] \arrow[d] \\ Spec(\frac{H}{\mathfrak{p}_S}) \arrow[uu, bend left, dotted] & Spec(k_H) \arrow[l] \arrow[uu, bend right, dotted] \arrow[u, dotted] \end{tikzcd}$$

Now $f \times \epsilon$, where ϵ is the zero section, restricted to $A_{\frac{H}{\mathfrak{p}_S}}[m]$ is the dotted morphism on the left. Since f is zero after the base change with k_H the dotted morphism on the left factors trough the diagonal. Since the diagram is commutative and the diagonal is open, also the morphism on the right factor trough the diagonal. Since $\frac{H}{\mathfrak{p}_S}$ is reduced, this is enough to conclude.
2)This is clear by 4.3.1, since the dimension of R_y, and hence of H, by [Sta16], Lemma 15.36.7. , is bigger or equal of 2 by hypothesis and the endomorphism of A_H are classified by a finite étale scheme over H. $\square$

With this we get that $f \in Im(EA(\pi_H)) \otimes \mathbb{Z}_l$. Now using again lemma 4.3.4, we find that f comes from an element in $End(A_{\overline{K}}) \otimes \mathbb{Z}_l$ and this conclude the proof.

4.3.2 Semi-simplicity

To prove semi-simplicity we will follow a specialization argument, based on a classical argument of Serre. The setting is the following. We have an abelian variety A of dimension g over a noetherian normal curve scheme over a field k, with generic fiber K, and we know that for every k-point s of R the induced representation of $\pi_1(k(s))$ over A_s is semisimple. We want to show that the representation induced on the generic fiber is semisimple. Recall that the map $\pi_1(K) \to \pi_1(R)$ is surjective, since it is the projection from $\pi_1(K)$ to the Galois of the maximal unramified extension of R. Denoting with Π_A the image of the representation induced on the fundamental group of some R algebra A we get the following commutative diagram:

$$\begin{tikzcd}
\pi_1(k(s)) \arrow[r] \arrow[d, "\rho_s"] & \pi_1(R) \arrow[d, "\rho_R"] & \pi_1(K) \arrow[l] \arrow[d] \\
\Pi_{k(s)} \arrow[r] \arrow[dr] & \Pi_R \arrow[d] & \Pi_K \arrow[l, "\simeq"'] \arrow[dl] \\
& GL_{2g}(\mathbb{Q}_l) &
\end{tikzcd}$$

So if we can find some $s \in Spec(R)$ such that the map $\Pi_{k(s)} \to \Pi_R$ is surjective we are done. To do this it's useful to recall the following, where we denote with $\Phi(G)$ the Frattini subgroup of any topological group G:

Lemma 4.3.6.
1)A map between profinite group $f : G \to H$ is surjective if and only if the map $G \to \frac{H}{\Phi(H)}$ is surjective.
2)$\Phi(\Pi_R)$ is open is Π_R.

Proof. 1)The image is contained in a maximal open subgroup M. Since the map is surjective on the quotient for the Frattini subgroup, $M\Phi(H) = H$. But $\Phi(H) \subseteq M$ so that $M = H$.
2)This follows from the fact that the image is is a compact l-adic Lie group, thanks to Cartan's theorem. But in any compact l-adic Lie group the Frattini is open, see [Ser97b], Page 148-149. $\square$

Thanks to the previous lemma $\rho^{-1}(\Phi(\Pi_R))$ is open and hence it gives us an Galois étale cover $X \to S$ with $Gal(X, S) = \frac{\Pi_1(R)}{\rho^{-1}(\Phi(\Pi_R))}$, where $S = Spec(R)$. Always thanks to the previous lemma, we just need to find an $s \in S$ such that the map $\pi_1(s) \to \frac{\Pi_R}{\Phi(\Pi_R)}$ is surjective. What is the image of $\pi_1(s) \to \pi_1(S)$? In general it is not clear, but for our purposes it is enough the following

Lemma 4.3.7. *$\pi_1(s)$ is contained in an open subgroup V of $\pi(S)$ if and only if the map $k(s) \to S$ lift to a S-map $k(s) \to X_V$ where X_V is the covering associated to V.*

Proof. If there exists the lifting then we are done thanks to the following commutative diagram:

$$\begin{tikzcd}
& \pi_1(X_V) \arrow[d] \\
\pi_1(s) \arrow[ur] \arrow[r] & \pi_1(S)
\end{tikzcd}$$

For the other implication observe that the connected components of the fiber over $k(s)$ are in bijection with the orbit of the action of $\pi(s)$ over $\frac{\pi_1(S)}{\pi_1(X_V)}$ and the degree of a component is the cardinality of the orbit. Since $\pi_1(s) \subseteq \pi_(X_V)$ we get that there exists at least one orbit with one element. $\square$

In particular we just need to find a rational point s such that for every covering X_U that correspond to an open subgroup of $\pi_1(S)$ containing $\rho^{-1}(\Phi(\Pi_R))$, i.e to a covering between X and S, s does not lift to a $k(s)$- rational point of X_U. It is not totally clear that this exists and to construct one we have to work a bit. First of all we take any étale covering $f : S \to U$, where U is an open subset of some $\mathbb{P}^n$ and up to replace S with a non empty open subscheme. X is also a covering of U and we take its Galois closure $\hat{X}$. For every $u \in U$ and $s \in f^{-1}(u)$ we get the following commutative diagram.

$$\begin{tikzcd}
 & \pi_1(s) \arrow[dl] \arrow[r] & \pi_1(S) \arrow[dl] \arrow[ddl] \arrow[dd] \arrow[r] & \Pi \arrow[dd] \\
\pi_1(u) \arrow[r] & \pi_1(U) \arrow[dl, two heads] & & \\
Gal(\hat{X}|U) & Gal(\hat{X}|S) \arrow[l] \arrow[r, two heads] & Gal(X|S) = \frac{\pi_1(S)}{\rho^{-1}(\Phi(\Pi_R))} \arrow[r, two heads] & \frac{\Pi}{\Phi(\Pi)}
\end{tikzcd}$$

We define $U' = U(k) - \cup_M p_M(\frac{\hat{X}}{M}(k))$ where M is varying over the proper subgroup of $Gal(\hat{X}|S)$ and $p_M : \frac{\hat{X}}{M} \to U$. Thanks to the previous discussion, if we show that U' is non empty we won, since then we can take any $u \in U'(k)$ and any $s \in f^{-1}(u)$ so that, by construction, s does not lift to any rational point of $\frac{\hat{X}}{M}$.
To prove this we introduce, following Serre, the notion of thin set.

Definition 4.3.8. If k is a field, we say that a subset A of $\mathbb{P}^n(k)$ is thin if there exists an algebraic variety X, with a morphism without rational section and finite generic fiber $\pi : X \to \mathbb{P}^n$ such that $A \subseteq \pi(X(k))$.
We say that a field is Hilbertian if it $\mathbb{P}^n(k)$ is not thin for every $n \in N$.

So to conclude the proof it is enough to show that if k is a function field over a finite field then it is Hilbertian.

Remark. [Ser97b], Page 121. 1)Every finite union of thin set is a thin set.
2)If a field k is Hilbertian then $\frac{k^*}{(k^*)^n}$ is infinite.
3)Every thin set is contained in a union of set in the following form:
-$f(X(k)$, where $X \to \mathbb{P}^n$ is a dominant morphism of degree bigger of 2 and X is a geometrically irreducible algebraic k-variety
-$i(X(k))$, where $i : X \to \mathbb{P}^n$ is the inclusion of a sub variety.

Lemma 4.3.9. *An infinite field is Hilbertian if and only if $\mathbb{P}^1(k)$ is not thin.*

Proof. Suppose that $\mathbb{P}^n(k)$ is thin for some n. We can assume that there is an geometrically irreducible variety $X \to \mathbb{P}^n$ such that the map is surjective on the k points. Then we have that for all the line $L \subseteq \mathbb{P}^n$ the base change map $X_L \to L$ is surjective on k points. But, via Bertini theorem ([Jou83], Theorem 6.10), we can choose a line such that X_L is geometrically irreducible so that $\mathbb{P}^1(k)$ is thin. □

Lemma 4.3.10. *1) For every field and every transcendental element t, $k(t)$ is Hilbertian.*
2)A finite extension L of an Hilbertian field K is Hilbertian.

Proof. 1)This can be proved in a number of different ways. We will give a non elementary proof that is based on some strong version of the Bertini theorem.

- Suppose first that k is infinite. If $k(t)$ is thin there exist finitely many $\pi_(X_i(k(t)))$ such that all but finitely many elements in $k(t)$ are contained in their union Ω. Observe that, since $\mathbb{A}^1_{k(t)} \to \mathbb{A}^2_k$ is birational, these morphisms extend to dominant maps $Y_i \to \mathbb{A}^2_k$, with Y_i is geometrically irreducible, since the set of point such that Y_i is geometrically irreducible is constructible and contains the generic point. Using Bertini theorem, see [Jou83] Theorem 6.3.4, there exists an open subset U_i of $\mathbb{A}_2$ such that $Y_i \times L_{a,b}$ is geometrically irreducible for every $(a, b) \in U_i(k)$, where $L_{a,b}$ is the line $at + b$.
In particular $at + b$ does not lift to a $k(t)$ rational point on X, since Y_i is geometrically irreducible and hence $at + b \notin \pi_i(X_i(K(t)))$. Hence $\lambda \cap k \times k \subseteq \cup_i(\mathbb{A}^2 - U_i)(k))$ but $k \times k - \cup_i(\mathbb{A}^2 - U_i)(k))$ is infinite, a contradiction.

- Assume now that k is finite. If we denote with H_d the set of degree d hypersurfaces of A_k^2 and with S_d the subset of H_d made by the H such that $Y_i \times H$ is irreducible, we have, by [CB16] Corollary 1.7, that
$$Lim_{d \mapsto +\infty} \frac{|S_d|}{|H_d|} = a > 0$$
. Reasoning as before, we get that $\Omega \cap H_d \subseteq H_d - S_d$ and hence that
$$Limsup_{d \mapsto +\infty} \frac{H_d \cap \Lambda}{H_d} \leq 1 - a < 1$$
. But as $k(t) = \cup_d H_d$ and $k(t) - \Lambda$ is finite we should have
$$lim_{d \mapsto +\infty} \frac{H_d \cap \Lambda}{H_d} = 1$$
.

2)Suppose $\mathbb{A}^n(L)$ is thin, i.e that there exists a morphism $X \to \mathbb{A}_L^n$ over L, such that $\mathbb{A}^n(L) = \pi(X(L))$, where X is an algebraic variety that we can assume affine. Then we apply the functor $Res_{L/K}$ to get a morphism $Res_{L/K}(X) \to Res_{L/K}(\mathbb{A}_L^n)$. Moreover we have the counit $\mathbb{A}_K^n \to Res_{L/K}(\mathbb{A}_L^n) = Res_{L/K}(\mathbb{A}_K^n \otimes L)$ that on K point is just the natural inclusion of K^n in L^n. Consider the following Cartesian diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"] & Res_{L/K}(X) \arrow[d] \\ \mathbb{A}_k^n \arrow[r] & Res_{L/K}(\mathbb{A}_L^n) \end{tikzcd}$$

Observe that $f(Y(K)) = \mathbb{A}^n(K)$ so that to conclude we have only to show that f has no rational section. But if f has a rational section s the universal property of Res would give us a rational section of π and this is not possible by assumption. $\square$

This last lemma and the work done in the previous sections give us:

Theorem 4.3.11. *The Tate conjecture it true for every function field of positive characteristic different from 2.*

Remark. Using a result of Deligne, the homotopy exact sequence for the fundamental group and some properties of algebraic groups, is possible to reduce the semisimplicity even to finite field k. Indeed suppose A is an abelian variety over a normal geometrically connected curve X defined over k and choose a point x of the curve defined over the $\overline{k}$. Then we have an exact sequence:

$$0 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \pi_1(x) = \Gamma_k \to 0$$

Then the action of the last one on the Tate module of A is semisimple, thanks to the Tate conjecture over finite fields. The result of Deligne, [Del80] Corollary 3.4.13, tells us that also the action of $\pi_1(X_{\overline{k}})$ is semisimple, and hence we get semisimplicity of the action of $\pi_1(X)$.

Chapter 5

Proof over number fields

The proof over number fields is similar in the spirit to the one over function fields, but the Diophantine step is more involved. The key problem is the presence of Archimedean valuations, that does not make possible the prove directly the boundedness of the height of a family of isogenous abelian varieties. To prove this boundedness we introduce another notion of height, the Faltings height, that takes care also of the infinite valuations. The first two sections are devoted to define and prove boundedness of this height. Next, using some Arkelov geometry, we will show that the boundedness of this height implies the one of the delta height.

5.1 Behavior of the Faltings height under l power isogenies

The aim of the following two sections is to prove the following theorem:

Theorem 5.1.1. *Let K be a number field, A an abelian variety over K with semistable reduction and G a sub l-divisible group of $A[l^\infty]$. Define $B_n := \frac{A}{G_n}$ and let $h(B_n)$ be the Faltings heights of B_n. Then the set $\{h_F(B_n)\}$ is finite.*

The proof relies in a careful study of the kernel of the isogenies $A \to B_n$ and some results about the representation $T_l(G)$.
For all this section K is a number field, $\mathcal{O}_K$ its ring of integer, v is a (finite) prime over l, A an abelian variety over K of dimension g with semistable reduction, G a sub l-divisible group of $A[l^\infty]$, of height h, and $B_n := \frac{A}{G_n}$. Moreover let $\mathcal{A}$ be the connected component of the Neron model of A

Definition 5.1.2. We define the Faltings heights of A as

$$h_F(A) := \frac{1}{[K:\mathbb{Q}]}\Big(log\big(\Big|\frac{s^*\Omega^g_{\mathcal{A}/\mathcal{O}_K}}{\omega s^*\Omega^g_{\mathcal{A}/\mathcal{O}_K}}\Big|\big) + \sum_{i:K\to\mathbb{C}} log\big(\Big| \int_{i(A)(\mathbb{C})} \omega \wedge \overline{\omega}\ \Big|^{-1}\big)\Big)$$

where $\omega \in \Omega^g_{\mathcal{A}/\mathcal{O}_K}$ is any non zero element.

Remark. 1)Using the product formula and the fact that $\Big|\frac{s^*\Omega^g_{\mathcal{A}/\mathcal{O}_K}}{\omega s^*\Omega^g_{\mathcal{A}/\mathcal{O}_K}}\Big| = \sum_v \Big|\frac{s^*\Omega^g_{\mathcal{A}_v/\mathcal{O}_{K,v}}}{\omega s^*\Omega^g_{\mathcal{A}_v/\mathcal{O}_{K,v}}}\Big|$ one shows that the Faltings height does not depend on the choice of ω.
2)If A has semistable reduction the Faltings height is stable by finite base change thanks to A.1.34 and the factor $\frac{1}{[K:\mathbb{Q}]}$.
3)If $s^*\Omega^g_{\mathcal{A}/\mathcal{O}_K}$ is principal then

$$h_F(A) = \frac{1}{[K:\mathbb{Q}]}\Big(\sum_{i:K\to\mathbb{C}} log\big(\Big| \int_{i(A)(\mathbb{C})} \omega \wedge \overline{\omega}\ \Big|^{-1}\big)\Big)$$

where $\omega \in \Omega^g_{\mathcal{A}/\mathcal{O}_K}$ is any generator of the module.

Denote with ψ_n the isogeny $A \to B_n$ of degree l^{nh}. We have an exact sequence $0 \to G_n \to A \to B_n \to 0$ over K so that, over $\mathcal{O}_K$ we have $0 \to \mathcal{G}_n \to \mathcal{A} \to \mathcal{B}_n \to 0$, where $\mathcal{G}_n$ is a quasi-finite flat group scheme, just looking at each fiber. We want to understand $h_F(A) - h_F(B_n)$ and we start observing that $\psi^*\omega_{B_n} = a\omega_A$ for some a in $\mathcal{O}_K$. Now we compute:

$$h_F(A) - h_F(B_n) = \frac{1}{[K:\mathbb{Q}]}\sum_{i:K\to\mathbb{C}} log(\frac{|\int_{i(A)(\mathbb{C})}\omega_A \wedge \overline{\omega_A}|^{-1})}{|\int_{B_n(\mathbb{C})}\omega_{B_n}\wedge\overline{\omega_{B_n}}|^{-1})} =$$

$$= \frac{1}{[K:\mathbb{Q}]}\sum_{i:K\to\mathbb{C}} log(\frac{|\int_{\mathbb{C}/\Lambda_A}\omega_A \wedge \overline{\omega_A}|^{-1})}{|\int_{\mathbb{C}/\Lambda_{B_n}}\omega_{i(B_n)}\wedge\overline{\omega_{B_n}}|^{-1})} = \frac{1}{[K:\mathbb{Q}]}\sum_{i:K\to\mathbb{C}} log(\frac{i(a)\overline{i(a)}|\int_{\mathbb{C}/\Lambda_A}\omega_A \wedge \overline{\omega_A}|^{-1})}{Deg(\psi_n)|\int_{\mathbb{C}/\Lambda_A)}\omega_A\wedge\overline{\omega_A}|^{-1})} =$$

$$= \frac{1}{[K:\mathbb{Q}]}\sum_{i:K\to\mathbb{C}} log(i(a)\overline{i(a)}) - \frac{1}{[K:\mathbb{Q}]}\sum_{i:K\to\mathbb{C}} log(Deg(\psi_n)) =$$

$$= \frac{2}{[K:\mathbb{Q}]}log(|N_K(a)|) - log(Deg(\psi_n))$$

where $i(A)(\mathbb{C}) = \mathbb{C}/\Lambda_A$, the same for B_n and we have used the formula for changing variables in the integral and the fact that ψ_n identifies Λ_A with a sub lattice of Λ_{B_n}, see A.1.1.

So we have that $h(B_n) = h(A)$ if and only if, remembering that $deg(\psi_n) = l^{nh}$, $l^{\frac{hn[K:\mathbb{Q}]}{2}} = |\mathcal{O}_K/a\mathcal{O}_K|$. Then we note that $\mathcal{O}_K/a\mathcal{O}_K$ is the coker of the map $s^*\Omega^g_{\mathcal{B}_n/\mathcal{O}_K} \to s^*\Omega^g_{\mathcal{A}/\mathcal{O}_K}$, i.e $s^*\Omega^g_{\mathcal{A}/\mathcal{B}_n}$. Moreover observe that, denoting with i_n the inclusion $\mathcal{G}_n \to \mathcal{A}$ and observing that $\mathcal{G}_n$ is the base change of $\mathcal{A} \to \mathcal{B}$ along the zero section, $s^*\Omega_{\mathcal{A}/\mathcal{B}} \simeq s^*i_n^*\Omega_{\mathcal{A}/\mathcal{B}} \simeq s^*\Omega_{\mathcal{G}/\mathcal{O}_K}$ so that $s^*\Omega_{\mathcal{A}/\mathcal{B}}$ is torsion and finitely generated and hence, it is isomorphic to $\oplus_{\mathfrak{p}\in Spec(\mathcal{O}_k)}(s^*\Omega_{\mathcal{A}/\mathcal{B}})_{\mathfrak{p}}$. A simple computation, using the fact that on every localization of a Dedekind domain we can diagonalize matrix, shows that

$$|coker(s^*\Omega^g_{\mathcal{B}_n/\mathcal{O}_K} \to s^*\Omega^g_{\mathcal{A}/\mathcal{O}_K})| = |coker(s^*\Omega_{\mathcal{B}_n/\mathcal{O}_K} \to s^*\Omega_{\mathcal{A}/\mathcal{O}_K}| = |s^*\Omega_{\mathcal{A}/\mathcal{B}_n}| = |s^*\Omega_{\mathcal{G}_n/\mathcal{O}_K}|$$

In conclusion we have shown the following:

$$h(A) = h(B_n) \text{ if and only if } l^{\frac{hn[K:\mathbb{Q}]}{2}} = |s^*\Omega_{\mathcal{G}_n/\mathcal{O}_K}|$$

Since $s^*\Omega_{\mathcal{G}_{n,v}/\mathcal{O}_K}$ is killed by a power of l its support is contained in the prime over l and hence, since it is finite, $s^*\Omega_{\mathcal{G}_{n,v}/\mathcal{O}_K} \simeq \prod_{v|l} s^*\Omega_{\mathcal{G}_{n,v}/\mathcal{O}_{K_v}}$ where $\mathcal{O}_{K_v}$ is the completion at v. Now $\mathcal{O}_{K_v}$ is a complete DVR and $\mathcal{G}_v$ is quasi finite, so that, thanks to A.4.4, $\mathcal{G}_{n,v} \simeq \mathcal{G}^f_{n,v}\coprod \mathcal{G}^\eta_{n,v}$ where the first factor it is a (universal) finite group scheme over $\mathcal{O}_{K_v}$. Since $h : \mathcal{G}^f_{n,v} \to \mathcal{G}_{n,v}$ is a group map and an open immersion, $s^*\Omega_{\mathcal{G}_{n,v}/\mathcal{O}_{K_v}} \simeq s^*h^*\Omega_{\mathcal{G}_{n,v}/\mathcal{O}_{K_v}} \simeq s^*\Omega_{\mathcal{G}^f_{n,v}/\mathcal{O}_{K_v}}$ so that $|s^*\Omega_{\mathcal{G}_{n,v}/\mathcal{O}_{K_v}}| = |s^*\Omega_{\mathcal{G}^f_{n,v}/\mathcal{O}_{K_v}}|$.
The natural question now is if the family $\mathcal{G}^f_{n,v}$ form a l-divisible group over $\mathcal{O}_{K_v}$. In general the answer is no, but the next lemma shows that is true up to replacing A with some $\mathcal{B}_n$.

Lemma 5.1.3. *There exists an $N >> 0$ such that $\frac{\mathcal{G}_{n+N,v}}{\mathcal{G}_{N,v}}$ for an l divisible group.*

Proof. Observe that the generic fiber of $\mathcal{G}_{N,v}$ form an l-divisible group. This follows from the fact that the intersection of l-divisible group is l-divisible over an a characteristic zero field, because the intersection of $\mathbb{Z}_p$ Γ_K invariant modules is a $\mathbb{Z}_p$ Γ_K invariant module, and from the equality $\mathcal{G}^f_{n,v} = \mathcal{G}^f_{n,v} \cap \mathcal{A}_v[l^n]$. To justify this equality, we observe that just by functoriality we have the inclusion from the left to the right and we have just to check that they have the same special fiber (and this is clear since only the finite part persist) and the same generic fiber. For the last, one check that they have the same K' point for every finite extension of K by the finiteness of $\mathcal{G}^f_{n,v}$.
Now the lemma follows from the following:

Claim If the generic fiber of a family of finite flat groups $G_n \to G_{n+1}$ form an l-divisible group over a discrete valuation ring R with fraction field K, then there exists an $N >> 0$ such that $\frac{G_{n+N}}{G_N}$ form an l-divisible group.

Proof. Observe that G_{i+1}/G_i is finite, flat and killed by p (since the map induced on the *Hom* on the generic fiber is injective) so that we have an homomorphism $G_{i+2}/G_{i+1} \to G_{i+1}/G_i$ that it is an isomorphism on the generic fiber. Thanks to this we have that the family G_{i+1}/G_i is a increasing chain inside the finite algebra $G_1\otimes_K$. Since the integral closure of R inside the above algebra is noetherian, there exists an N such that G_{i+1}/G_i is an isomorphism for all $i > N$. We claim that this N works i.e that $G'_n = G_{N+n}/G_N$ form an l-divisible group with the natural inclusion map. We have the following commutative diagram:

$$\begin{tikzcd} G_{N+v+1}/G_N \arrow[r, "p^v"] \arrow[d, "\alpha"] & G_{N+v+1}/G_N \\ G_{N+v+1}/G_{N+v} \arrow[r, "\beta"] & G_{N+1}/G_N \arrow[u, "\gamma"] \end{tikzcd}$$

where β is induced by the multiplication by p^v and it is an isomorphism. Then, since γ it is a closed immersion, $Ker(p^v) = Ker(\alpha) = G_{N+v}/G_N$ and hence we are done. $\square$

So, up to replacing A with B_N, we can assume that $\mathcal{G}^f_{n,v}$ is a l-divisible group. Observe that we have the exact sequence $0 \to (\mathcal{G}^f_{n,v})^0 \to \mathcal{G}^f_{n,v} \to (\mathcal{G}^f_{n,v})^{\text{ét}} \to 0$, where the first map it is an open immersion of group scheme. Reasoning as before we get that $s^*\Omega_{\mathcal{G}^f_{n,v}/\mathcal{O}_{K_v}} \simeq s^*\Omega_{(\mathcal{G}^f_{n,v})^0/\mathcal{O}_{K_v}}$. But this is computable! In fact observe that f $\mathcal{G}^0_{v,n} = Spec(A_{v,n})$ then $A_{v,n}$ is a connected finite algebra over $\mathcal{O}_{K,v}$ and hence it is generated by an algebraic element α with minimum polynomial f (tensoring with the residue field leaves $A_{v,n}$ connected so that the tensor is generated by an element and hence (Nakayama) $A_{v,n}$ is generated by an element). So we get that $\Omega_{A_{v,n}/\mathcal{O}_{K,v}} \simeq \frac{A_{v,n}}{f'(\alpha)}$. But $\Omega_{A_{v,n}/\mathcal{O}_{K,v}} \simeq s^*\Omega_{(\mathcal{G}^f_{n,v})^0/\mathcal{O}_{K_v}} \otimes_{\mathcal{O}_K} A_{v,n}$, thanks to A.1.15, and $A_{v,n}$ is free of rank $Rank((\mathcal{G}^f_{n,v})^0)$ so that $|s^*\Omega_{(\mathcal{G}^f_{n,v})^0/\mathcal{O}_{K_v}}|^{\frac{1}{Rank((\mathcal{G}^f_{n,v})^0)}} = |\frac{A_{v,n}}{f'(\alpha)}|$. But $|\frac{A_{v,n}}{f'(\alpha)}| = |\frac{\mathcal{O}_{K,v}}{N_K(f'(\alpha))}| = |\frac{\mathcal{O}_{K,v}}{Disc_{\mathcal{O}_{K,v}}((\mathcal{G}^f_{n,v})^0)}|$.
To resume, we are reduced to show that

$$l^{\frac{hn[K:\mathbb{Q}]}{2}} = \prod_{v|l} \left| \frac{\mathcal{O}_{K,v}}{Disc_{\mathcal{O}_{K,v}}((\mathcal{G}^f_{n,v})^0)} \right|^{\frac{1}{Rank(\mathcal{G}^0_{v,n})}}$$

For this we have the following result of Tate, remembering that we can assume that the $\mathcal{G}^f_{n,v}$ form an l-divisible group.

Proposition 5.1.4. *If G is a p-divisible group of height h and with associated formal group of dimension n then $Disc(G_v) = (p^{nvRank(G_v)})$*

Proof. See [Lie00] Prop 6.2.12 $\square$

So we get

$$\prod_{v|l} |\frac{\mathcal{O}_{K,v}}{Disc_{(\mathcal{G}^f_{n,v})^0)}}|^{\frac{1}{Rank((\mathcal{G}^f_{n,v})^0))}} = \prod_{v|l} |\frac{\mathcal{O}_{K,v}}{(l^{d_v n Rank((\mathcal{G}^f_{n,v})^0)})})|^{\frac{1}{Rank((\mathcal{G}^f_{n,v})^0))}} = \prod_{v|l} l^{d_v n[K_v:\mathbb{Q}_l]}$$

where d_v is the dimension of the formal group associated to the l divisible group $(\mathcal{G}^f_{n,v})^0$. So we are left to show that

$$\sum d_v[K_v : \mathbb{Q}_l] = \frac{1}{2}h[K : \mathbb{Q}]$$

.
The insight of Faltings is that this two numbers are the Hodge-Tate weight of the same character, see A.4.6. Define $W = T_l(G)$ and $V = Ind^{\Gamma_K}_{\Gamma_\mathbb{Q}}(W)$. We will first show that $Det(V)_{|\mathbb{Q}_l}$ is Hodge-Tate of weight $\sum d_v[K_v : \mathbb{Q}_l]$ and then that it is Hodge-Tate of weight $\frac{1}{2}h[K : \mathbb{Q}]$.
First of all, observe that $V_{|\mathbb{Q}_l} = (Ind^{\Gamma_K}_{\Gamma_\mathbb{Q}}(W))_{|\mathbb{Q}_l} = \oplus_{s\in\Gamma_{\mathbb{Q}_l}\backslash\Gamma_\mathbb{Q}/\Gamma_K} Ind^{(\Gamma_K)_s}_{\Gamma_\mathbb{Q}}(W_s)$, thanks to A.4.7, where $(\Gamma_K)s$ is $s\Gamma_K s^{-1} \cap \Gamma_{\mathbb{Q}_l}$ and W_s is the representation of $(\Gamma_K)_s$ given by $p(x) = p(s^{-1}xs)$. Since K is Galois we get $V_{|\mathbb{Q}_l} = \oplus_{s\in\Gamma_{\mathbb{Q}_l}\backslash\Gamma_\mathbb{Q}/\Gamma_K} Ind^{\Gamma_{K_v}}_{\Gamma_{\mathbb{Q}_l}}(W_s)$ for some v over l. Since $\Gamma_{\mathbb{Q}_l}\backslash\Gamma_\mathbb{Q}/\Gamma_K$ is the set of embedding of K in $\overline{\mathbb{Q}}_l$, i.e the set of primes v over l, we get that $V_{|\mathbb{Q}_l} = \oplus_{v|l} Ind^{\Gamma_{K_v}}_{\Gamma_{\mathbb{Q}_l}}(W_{|K_v})$. So $Det(V_{|\mathbb{Q}_l}) = \otimes_{v|l} Det(Ind^{\Gamma_{K_v}}_{\Gamma_{\mathbb{Q}_l}}(W_{|K_v}))$. Observe that over a sufficiently big finite extension this representations are isomorphic to $Det(W_{|K_v})^{[K_v:\mathbb{Q}_l]}$. Since the Hodge-Tate propriety is insensitive of finite extension, A.4.6, we get that $Det(V_{|\mathbb{Q}_l})$ is Hodge-Tate of weight $\sum d_v[K_v : \mathbb{Q}_l]$ thanks to the following (highly non trivial) theorem, that we will prove in the next section:

Theorem 5.1.5. *$Det(W_{|K_v})$ is Hodge-Tate d_v.*

Now that we know that $Det(V)$ is hodge of some weight d at l we can do another computation. Observe that this representation is an l-adic character χ of $(\Gamma_{\mathbb{Q}})^{ab}$ unramified outside finitely many primes. Define $\chi_0 = \chi_l^{-d}\chi$ where χ_l is the cyclotomic character.
Then χ_0 is of finite order. In fact, thanks to Kronecker-Weber theorem, $(\Gamma_{\mathbb{Q}})^{ab} \simeq \prod_p \mathbb{Z}_p^*$ where each $\mathbb{Z}_p$ is the inertia group at p. Since χ is unramified at almost every prime, χ_0 is a map $\mathbb{Z}_{p_1}^*, ..., \times \mathbb{Z}_{p_n}^* \times \mathbb{Z}_l^* \to \mathbb{Z}_l^*$. Observe that each $\mathbb{Z}_{p_i}^*$ has finite image (in fact it isomorphic to $F \times \mathbb{Z}_p$ with F finite and so in it trivial outside a finite set). Moreover since χ_0 is Hodge-Tate of weight 0 at l also $\mathbb{Z}_l^*$ has finite image since it is the inertia group at l (see A.4.6).
So we can write $\chi = \chi_l^d \chi_0$ with χ_0 of finite order. Then observe that for some Frobenius $|\chi(F_p)| = |\chi_l^d(F_p)| = p^d$. But, thanks to Weil conjectures A.4.8, the eigenvalue of the Frobenius acting on $V_l(A)$ and hence on $V_l(W)$ have complex absolute value $p^{1/2}$. But this are the same eigenvalue (choosing a Frobenius in Γ_K at same place of good reduction) of V. So the complex absolute value of the character of $Det(V)$ at this Frobenius is $p^{[K:\mathbb{Q}]h/2}$, since V is a representation of dimension mh.

5.2 $Det(W_{|K_v})$ is Hodge-Tate of weight d_v

In this section we will proof 5.1.5. The theorem is corollary of the following two theorems.

Theorem 5.2.1 (Hodge-Tate decomposition). *[Tat67] Let K be a p-adic field, G a p-divisible group over $\mathcal{O}_K$, t_G the tangent space of the formal group associated to it with dual t_G^* and $t_G(\mathbb{C}_K) = t_G \otimes \mathbb{C}_K$. Then $Hom(T(G), \mathbb{C}_K) \simeq t'_G(\mathbb{C}_K) \oplus t_G(\mathbb{C}_K)(-1)$ and so, taking duals, $T(G) \simeq t_G(\mathbb{C}_K)(1) \oplus t*_{G'}(\mathbb{C}_K)$*

Theorem 5.2.2 (Orthogonality theorem). *With the same notation of the previous theorem assume further that G is the p-divisible group of the p torsion of a semistable abelian variety. Then, denoting $T_l(G)_f$ the p divisible group associated to the finite part of G, $\frac{T_l(G)}{T_l(G)_f}$ is an unramified representation of Γ_K.*

Proof. of thm 5.1.5.
We have an exact sequence, where $W_f \subseteq W$ is the submodule induced by the finite part of G, $0 \to W_f \to W \to \frac{W}{W_f} \to 0$ and hence an isomorphism $Det(W)_{|\mathbb{Q}_l} \simeq Det(W_f)_{|\mathbb{Q}_l} \otimes Det(\frac{W}{W_f})_{|\mathbb{Q}_l}$. Now W_f is a representation coming from a l-divisible group of $\mathcal{O}_{K,v}$ and so, by 5.2.1, $Det(W_f)_{|\mathbb{Q}_l}$ it is Hodge-Tate of weight d_v. To conclude, we observe that, by 5.2.2, $Det(\frac{W}{W_f})_{|\mathbb{Q}_l}$ is unramified and hence, thanks to A.4.6, it is Hodge-Tate of weight 0. □

5.2.1 Hodge-Tate decomposition

We start working over a p-adic field K, with ring of integers $\mathcal{O}_K$, absolute Galois group Γ_K and we fix a p-divisible group G over $\mathcal{O}_K$, with dual G'. The aim of this section is to show that $Hom(T(G), \mathbb{C}_K) \simeq t_G(\mathbb{C}_K) \oplus t_{G'}^*(\mathbb{C}_K)(-1)$ as Galois module. Let $D \in \{K, \mathbb{C}_K\}$. We will denote $\mathfrak{m}_D$ the maximal ideal of D and $U_D = 1 + \mathfrak{m}_D$. We start observing the following dualities:

- $\Phi(G) \simeq Hom(T(G'), \mu_{p^\infty})$
- $T(G) \simeq Hom(T(G), \mathbb{Z}_p(1))$
- $T(G') \simeq Hom_D(G_D, \mathbb{G}_m(p))$ where G_D is the base change of G on D and the Hom are the homomorphism as p-divisible group

Observe that the second duality give us for every element of $T(G')$ a family of maps $\psi_{v,B} : G_v(B) \to \mathbb{G}_m(p)_v(B) = \mu_{p^v}(B)$ where v is an integer and B is a D algebra. The key observation is the following lemma:

Lemma 5.2.3. $U_{\mathbb{C}_K} \simeq \varprojlim_i \varinjlim_v \mathbb{G}_m(p)_v(\frac{\mathbb{C}_K}{\mathfrak{m}_{\mathcal{O}_K}^i \mathbb{C}_K})$

Proof. See [Ntls10] L10, Example 2.2 □

So, to put all together the maps given by duality, we define $G(D)$ as $\varprojlim_i \varinjlim_v G_v(\frac{D}{\mathfrak{m}_R^i D})$. We get a map $T(G') \times G(D) \to \mathbb{G}_m(p)(D) \simeq U$ and hence a map $G(D) \to Hom(T(G'), U)$. Observe that we have an exact sequence $0 \to \mu_{p^\infty} \to U \to \mathbb{C}_K \to 0$, where the last map is the logarithm. This gives us an exact sequence $0 \to Hom(T(G'), \mu_{p^\infty}) \to Hom(T(G'), U) \to Hom(T(G'), \mathbb{C}_K) \to 0$. We want to extend this sequences to any G.

Lemma 5.2.4.
- *$G^0(D) \simeq Hom_{cont}(\mathcal{A}, D) \simeq \mathfrak{m}_D^{Dim(G)}$*
- *$G^{\acute{e}t}(D) = \varinjlim_v G_v^{\acute{e}t}(\frac{D}{\mathfrak{m}_{\mathcal{O}_K}})$ and hence it is torsion.*
- *The sequences $0 \to G^0(D) \to G(D) \to G^{\acute{e}t}(D) \to 0$ is exact.*
- *$G(\mathcal{O}_{\mathbb{C}_K})_{tors} = \Phi(G)$*

Proof. See [SC86] Chapter 3,6 □

Thanks to the first point of the lemma we can define a functorial continuous homomorphism $log : G^0(D) \to t_G(Frac(D))$ and we recall that it is a local isomorphism in a neighborhood of the identity, see A.4.2. Using the second point we can extend the extend the map to the whole $G(D)$, since for every x we can chose an n such that $p^n x \in G^0(D)$ and then define $log(x) = log(p^n x)$.

Lemma 5.2.5. *We have an exact sequence $0 \to \Phi(G) \to G(\mathcal{O}_{\mathbb{C}_K}) \to t_G(\mathbb{C}_K) \to 0$ and $log(G(\mathcal{O}_K))$ spans $t_G(K)$ as $\mathbb{Q}_p$ vector space.*

Proof. Recall that the torsion of $G(\mathcal{O}_{\mathbb{C}_K})$ is equal to $\Phi(G)$ and it is contained in the kernel of log, since $t_G(\mathbb{C}_K)$ is torsion free. Moreover if x is in the kernel then $log(p^n x) = 0$ for some n. But log is an isomorphism in a neighborhood of the identity, so, up to choosing n big enough, $log(p^n x) = 0$ implies $p^n x = 0$ i.e x is torsion.
For the surjectivity observe that the image of the logarithm restricted to some some open U is an open subgroup containing the identity of $t_G(\mathbb{C}_K)$ so that $log(\mathcal{O}_{\mathbb{C}_K})$ contains an open subgroup containing the identity. But then, for every $x \in t_G(\mathbb{C}_K)$, using A.4.1, there exists n such that $p^n x \in log(\mathcal{O}_{\mathbb{C}_K})$ i.e the cocker of the log is torsion so that $log(\mathcal{O}_{\mathbb{C}_K}) \otimes \mathbb{Q}_p = t_G(\mathbb{C}_K)$. Observe that this reasoning applied with $\mathcal{O}_K$ give us the last statement. To conclude it is enough to show that $G(\mathcal{O}_{\mathbb{C}_K})$ is divisible since this implies that $log(\mathcal{O}_{\mathbb{C}_K}) \otimes \mathbb{Q}_p \simeq log(\mathcal{O}_{\mathbb{C}_K})$.
To show this, thanks to the previous lemma, it enough to work with $G^0(\mathcal{O}_{\mathbb{C}_K})$ and $G^{\text{ét}}(\mathcal{O}_{\mathbb{C}_K})$. For the first observe that this is true thanks to the fact that the multiplication by p in the formal group is finite and faithful flat and the isomorphism $G^0(\mathcal{O}_{\mathbb{C}_K}) \simeq Hom_{cont}(\mathcal{A}, \mathcal{O}_{\mathbb{C}_K})$. For the second we observe that this is clear since $G^{\text{ét}}(\mathcal{O}_{\mathbb{C}_K}) = (\frac{\mathbb{Q}_p}{\mathbb{Z}_p})^n$ for some n (thanks to Hensel lemma and the fact that the residue field is algebraically closed). □

Putting all together we get the following commutative diagram with exact rows and Γ_K invariant vertical maps:

$$\begin{tikzcd}
0 \arrow[r] & \Phi(G) \arrow[r] \arrow[d, "\alpha_0"] & G(D) \arrow[r] \arrow[d, "\alpha"] & t_G(\mathbb{C}_K) \arrow[r] \arrow[d, "d\alpha"] & 0 \\
0 \arrow[r] & Hom(T(G'), \mu_{p^\infty}) \arrow[r] & Hom(T(G'), U) \arrow[r] & Hom(T(G'), \mathbb{C}_K) \arrow[r] & 0
\end{tikzcd}$$

Remark. α_0 is bijective, thanks to the duality at the beginning of the section. As a consequence $ker(\alpha) \simeq ker(d\alpha)$ and hence $ker(\alpha)$ is a $\mathbb{Q}_p$ vector space.

Proposition 5.2.6.
1) α and $d\alpha$ are injective
2) $\alpha_{\mathcal{O}_K} : G(\mathcal{O}_K) \to Hom_{\Gamma_k}(T(G'), U_{\mathbb{C}_K})$ and $d\alpha_{\mathcal{O}_K} : Hom_{\Gamma_k}(T(G'), \mathbb{C}_K)$ are bijective.

Proof. 1. • It is enough to show that α is injective when restricted to $G(\mathcal{O}_K)$.
Indeed, if we know this we also know, by the previous remark, that it is injective on $log(G(R))$, that spans $t_G(K)$ as $\mathbb{Q}_p$ vector space thanks to 5.2.5. But then we can factorize $d\alpha$ in the following way: $t_G(\mathbb{C}_K) \simeq t_G(K) \otimes \mathbb{C}_K \to Hom_{\Gamma_K}(T(G'), \mathbb{C}_K) \otimes \mathbb{C}_K \to Hom(T(G'), \mathbb{C}_K)$ so what want follows from the following:

Claim For every $\mathbb{C}_K$ vector space with a semi linear action of Γ_K the map $W^{\Gamma_K} \otimes \mathbb{C}_K \to W$ is injective.

Proof. Suppose that $e_i \in W^{\Gamma_K}$ are linearly independent and take the shortest relation $\sum a_i e_i = 0$ in W with $a_1 = 1$. Then, since e_i is invariant, we get for every $\sigma \in \Gamma_k$

$$\sum(\sigma(a_i) - a_i)e_i = 0$$

Since $a_1 = 1$, we get $\sigma(a_1) = a_1$ and hence a shorter relation. As a consequence, $\sigma(a_i) = a_i$ for every i and hence, using A.4.6, $a_i \in K$, a contradiction. $\square$

- $Ker(\alpha) \cap G(\mathcal{O}_K)$ is a vector space.
For this it enough to show that $G(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K} = G(\mathcal{O}_K)$, since taking invariants in left exact and send vector spaces to vector spaces. Taking invariant of the exact sequence $0 \to G^0(\mathcal{O}_{\mathbb{C}_K}) \to G(\mathcal{O}_{\mathbb{C}_K}) \to G^{\text{ét}}(\mathcal{O}_{\mathbb{C}_K}) \to 0$, we get get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & G^0(\mathcal{O}_K) \arrow[r] \arrow[d] & G(\mathcal{O}_K) \arrow[r] \arrow[d] & G^{\text{ét}}(\mathcal{O}_K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G^0(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K} \arrow[r] & G(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K} \arrow[r] & G^{\text{ét}}(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K} &
\end{tikzcd}$$

Now the last vertical map is an isomorphism and also the second thanks to the fact that

$$G^0(\mathcal{O}_K) = \mathfrak{m}^n_{\mathcal{O}_K} = (\mathfrak{m}^n_{CC_K})^{\Gamma_K} = G^0(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K}$$

using again A.4.6 and 5.2.4. The snake lemma gives us the result.

- Now we show that it is injective on $G(\mathcal{O}_K)$. We just need to show that $G^0(\mathcal{O}_K) \cap ker(\alpha_{\mathcal{O}_K}) = 0$. Indeed, if we know this we are done since $ker(\alpha) \cap \mathcal{O}_K$ is torsion free (is a $\mathbb{Q}_p$ vector space) e $G^{\text{ét}}(\mathcal{O}_K)$ is torsion thanks to 5.2.4. Now observe that we have a commutative diagram with injective vertical maps:

$$\begin{tikzcd}
G^0(\mathcal{O}_K) \arrow[r, "\alpha^0"] \arrow[d] & Hom(T((G^0)'), \mathbb{C}_K) \arrow[d] \\
G(\mathcal{O}_K) \arrow[r, "\alpha"] & Hom(T(G'), \mathbb{C}_K)
\end{tikzcd}$$

So we see that $G^0(\mathcal{O}_K) \cap ker(\alpha_{\mathcal{O}_K}) = G^0(\mathcal{O}_K) \cap ker(\alpha^0)$. So we can assume that G is connected and we have to show that $ker(\alpha) \cap G(\mathcal{O}_K)$ is zero. $ker(\alpha)$ is a vector space and hence it is divisible. But $G(\mathcal{O}_K) = \mathfrak{m}^n_{\mathcal{O}_K}$ and, since the valuation on $\mathcal{O}_K$ is discrete, all the divisible submodules of $\mathfrak{m}^n_{\mathcal{O}_K}$ are trivial.

2.
- We know that the map are injective by the previous point. First we show that it is enough to prove that $coker(d\alpha_{\mathcal{O}_K}) = 0$. In fact, by left exactness of the fixed point functor, we have $coker(d\alpha_{\mathcal{O}_K}) \subseteq coker(d\alpha)^{\Gamma_K}$ and $coker(\alpha_{\mathcal{O}_K}) \subseteq coker(\alpha)^{\Gamma_K}$.
Moreover $coker(\alpha)^{\Gamma_K} \simeq coker(d\alpha)^{\Gamma_K}$, so that $coker(\alpha_{\mathcal{O}_K}) \subseteq coker(d\alpha_{\mathcal{O}_K})$ and so the claim follows.
- Since $cocker(d\alpha_{\mathcal{O}_K})$ is a K vector space, it is enough to show that

$$Dim(Hom_{\Gamma_K}(T(G'), \mathbb{C}_K) = Dim(G)$$

By injectivity of $d\alpha_{\mathcal{O}_K}$ and duality we have

$$Dim(Hom_{\Gamma_K}(T(G'), \mathbb{C}_K) \geq Dim(G) \text{ and } Dim(Hom_{\Gamma_K}(T(G), \mathbb{C}_K) \geq Dim(G')$$

To conclude the proof it is enough to prove the following two facts.
  - $Dim(Hom_{\Gamma_K}(T(G), \mathbb{C}_K) + Dim(Hom_{\Gamma_K}(T(G'), \mathbb{C}_K) \leq h$
  Observe that, by the duality explained at the beginning of the section, we have

$$Hom(T(G'), \mathbb{Z}_p)(-1) = Hom(T(G'), \mathbb{Z}_p(1)) = T(G)$$

  . Tensoring with $\mathbb{C}_K$ and $\mathbb{Z}_p(1)$ we get

$$Hom(T(G'), \mathbb{C}_K) = T(G) \otimes \mathbb{C}_K(1) = Hom(Hom(T(G), \mathbb{C}_K), \mathbb{C}_K(1))$$

. So we have a Γ_K invariant perfect paring

$$Hom(T(G'), \mathbb{C}_K) \times Hom(T(G), \mathbb{C}_K) \to \mathbb{C}_K(1)$$

Taking fixed points we get that $Hom_{\Gamma_K}(T(G), \mathbb{C}_K)$ and $Hom_{\Gamma_K}(T(G), \mathbb{C}_K)$ are orthogonal to each other, using A.4.6, and hence that

$$Hom_{\Gamma_K}(T(G), \mathbb{C}_K) \otimes \mathbb{C}_K \ Hom_{\Gamma_K}(T(G'), \mathbb{C}_K) \otimes \mathbb{C}_K$$

are orthogonal to each other. Thanks to claim done in the proof before, we have that are subspace of $Hom(T(G), \mathbb{C}_K)$ and $Hom(T(G'), \mathbb{C}_K)$ and so, by non degeneracy of the pairing

$$Dim(Hom_{\Gamma_K}(T(G'), \mathbb{C}_K)) + Dim(Hom_{\Gamma_K}(T(G), \mathbb{C}_K)) =$$
$$Dim(Hom_{\Gamma_K}(T(G'), \mathbb{C}_K) \otimes \mathbb{C}_K) + Dim(Hom_{\Gamma_K}(T(G), \mathbb{C}_K) \otimes \mathbb{C}_K) \leq$$
$$\leq DimHom(T(G), \mathbb{C}_K) = h$$

– $Dim(G)+Dim(G') = h$ Since all the number in play are stable by passing to the residue field and base extension, we can assume that G is defined over an algebraically closed field of characteristic p. We have the following commutative diagram of fppf sheaves with exact rows, where F is the Frobenius and V is the dual of the Frobenius, see A.1.23:

$$\begin{tikzcd}
0 \arrow[r] & Ker(F) \arrow[r] \arrow[d] & G \arrow[r, "F"] \arrow[d, "p"] & G^{(p)} \arrow[r] \arrow[d, "V"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & G \arrow[r, "id"] & G \arrow[r] & 0
\end{tikzcd}$$

The snake lemma give us an exact sequence $0 \to Ker(F) \to Ker(p) \to Ker(V) \to 0$. Observe that $Ker(p) = G_1$ and hence has order p^h. $Ker(V)$ is the dual of the cocker of the frobenius $G_1' \to (G_1^{(p)})'$. But this has the same order of the kernel of this map and so we are done if we show that the kernel of the frobenius has order $p^{dim(G)}$. We conclude observing that the frobenius is a finite map of rank $p^{dim(G)}$.

□

Proof. of 5.2.1 We have shown that there is a pairing such that $Hom(T(G), \mathbb{C}_K)$ and $Hom(T(G'), \mathbb{C}_K)$ are orthogonal to each other. Hence we have an exact sequence $0 \to t_{G'}(\mathbb{C}_K) \to Hom(T(G), \mathbb{C}_K) = Hom(Hom(T(G'), \mathbb{C}_K), \mathbb{C}_K(1)) \to Hom(t_G(\mathbb{C}_K), \mathbb{C}_K(1))$ By a dimension counting we get that the last map is surjective, and hence we have what we want using A.4.6. □

5.2.2 Orthogonality theorem

Let K be a p-adic field. We start with a semiabelian variety A, with connected component A^0, over $R = \mathcal{O}_K$ with reduction $0 \to C \to A_k^0 \to B \to 0$, dimension g, toric part C of dimension t and abelian part B of dimension a, so that $g = a + t$. Then we have that for every N, thanks to the snake lemma and the fact that the multiplication by N is surjective on the toric part, an exact sequence $0 \to C[N] \to A_k^0[N] \to B[N] \to 0$, so that $A_k^0[N]$ is finite of rank N^{t+2a}. Now $A[N]$ is flat and quasi finite, so that we can write $A[N] = A[N]_f \coprod A[N]_\eta$, with the first term finite group scheme with special fiber $A_k[N]$. Now the sequence of (open and closed) subgroups $C[N] \subseteq A_k^0[N]_f \subseteq A_k[N]$ over k lift to a sequence of finite flat group scheme over R,thanks to A.4.4, $A[N]_t \subseteq A[N]_f^0 \subseteq A[N]$ over R. If we put $N = p^n$ for n is varying, we get two p-divisible groups $G = \{A[p^\infty]_t\}$ and $H = \{A[p^\infty]_f^0\}$. Taking generic fiber, we get two p-sub divisible groups of $T_l(A)$, $T_l(A)_t$ and $T_l(A)_f$. Observe that the rank of the first one is t, while the rank of the second one is $2a + t$ and that, since it is true on the special fiber, the first one has étale Cartier dual. If G is a p-divisible group, in this section we will denote with $D(G)$ the Cartier dual of it. Recall that we want to show that $T_l(A)/T_l(A)_f$ is unramified as Γ_k representation so that it is enough to show $T_l(A)/T_l(A)_f \simeq D(T_l(A')_t)$, since étale representation of local fields are unramified (all the étale group schemes become constant after a finite unramified extension). To prove this we have just to show, thanks to some rank consideration ($T_l(A)$ has rank $2g$) that $T_l(A)_f$ and $T_l(A')_t$ annihilate each other under the Weil pairing. To prove this it is enough to prove that every map between $T_(A)_f$ and $D(T_l(A')_t)$ is zero. We do first some reductions and we start to show that it enough to prove that $Hom_R(H, D(G)) = 0$.

Lemma 5.2.7 (Main theorem of Tate on p-divisible group [Tat67]). *For every p-divisible group G, H, the natural map $Hom_R(G, H) \to Hom(G_K, H_K)$ is bijective.*

Proof. It is clearly injective and to prove that is it surjective we take a map $f : G_K \to H_K$. Consider p-divisible group graph $T(f) \subseteq T(G_K) \times T(H_K)$ and observe that it is a $\mathbb{Z}_p$ direct summand, since the quotient injects in $T(H_K)$ via the map $(x, y) \mapsto y - f(x)$ and $\mathbb{Z}_p$ is $P.I.D.$. We claim that there exists a p-divisible group $E \subseteq G \times H$ over R such that $T(E_K) \simeq T(f)$. If we can prove this we are done. In fact the natural map $E \to H$ is an isomorphism in the generic fiber and hence an isomorphism (it is enough to check that E and H have the same discriminant but thanks to 5.1.4 and 5.2.6 this is determined by the generic fiber) and then the composition of the inverse of this map with the natural map $G \to E$ does the work.
So we have just to prove the claim. Since it is a direct summand, $T(f)$ correspond to a sub p-divisible group E^* of the generic fiber and hence to a family of subgroup E^*_v. Then we can take E_v the closure of E^*_v in $G_v \times H_v$ and we get a family $\{E_v \to E_{v+1}\}$. Thanks to the claim in 5.1.3 for some w big enough $\frac{E_{w+v}}{E_w}$ form p-divisible group over R and they do the job. □

Now we pass from R to the residue field k thanks to the following

Lemma 5.2.8. *For every p-divisible group G, H, the natural map $Hom_R(G, H) \to Hom_k(G_k, H_k)$ is injective.*

Proof. We start with a map such that $f \otimes k = 0$ and we want to show that f is equal to zero. By induction we will show that $f \otimes \frac{R}{\mathfrak{m}^n} = 0$, this is enough since this shows that for every n and every v the augmentation ideal, the kernel of the zero section $\mathcal{O}(H) \to K$ is contained in $ker(f^*_v) + m^n$ and hence, thanks to Krull intersection, in $Ker(f_v)$. So we have to show that if $f \otimes \frac{R}{\mathfrak{m}^n} = 0$ then $f \otimes \frac{R}{\mathfrak{m}^{n+1}} = 0$. This is equivalent to show that if $f \otimes \frac{R}{\mathfrak{m}^{n+1}}/\mathfrak{m}^n = 0$ then $f \otimes \frac{R}{\mathfrak{m}^{n+1}} = 0$ so that we have to show that if I is an ideal of some complete local ring R killed by $\mathfrak{m}$ and $f \mod I = 0$ then $f = 0$. Moreover we can prove that $f \circ [p] = 0$ since $[p]$ is surjective in the category of fppf sheaves.
If G and H are étale then it is enough to observe that, since we are working over a complete ring, that taking special fiber is fully faithful. If G is connected then we go to formal groups, using A.1.25, and f is a map $R[[x_1, ..., x_n]] \to R[[x_1, ..., x_n]]$ such that $f(x_i)$ has coefficient in I (since $f \mod I = 0$) and zero constant term (since it must preserve the unit section). Then we have $[p] \circ f(x_i) = [p]h$where h has coefficient in I and no constant term so that it is zero since it $pI = 0$, $I^2 = 0$ and $[p]h$ looks like $\sum_i pa_i x_i + g$ where $a_i \in I$ and g as coefficient in I^2.
So, using connected étale sequence, we are reduced in the situation when G is étale , and hence, up to a finite base change, we can assume $G = \mathbb{Q}_p/\mathbb{Z}_p$, and H is connected. But then at every finite level v this is a map $\frac{\mathbb{Z}}{p^v\mathbb{Z}} \to H_n$ i.e. a family of element in $H_n(R^n)$ and $H_n = Spec(R[[x_1, ..., x_n]][p^v](x_1, ..., x_n))$. The map is zero mod I, so this elements go to zero in $H_n(R/I)$, so that the map is zero since every of this element is killed by p ($pI = 0$). □

Now we are almost done. Observe that the special fiber of G it $T[p^\infty]$ and that the special fiber of H is $A^0_k[p^\infty]$ so we are reduced to prove, thanks two the previous two lemmas, that $Hom(A^0_k[p^\infty], D(T[p^\infty])) = 0$. It is enough to prove that $Hom(T[p^\infty]), D(T[p^\infty])) = 0$ and $Hom(B[p^\infty], D(T[p^\infty])) = 0$. Up to a finite base change we can assume T split and then $T = \mathbb{G}_m$, so the statements become $Hom(\mu_{p^\infty}, \frac{\mathbb{Q}_p}{\mathbb{Z}_p}) = 0$ and $Hom(B[p^\infty], \frac{\mathbb{Q}_p}{\mathbb{Z}_p}) = 0$. $Hom(\mu_{p^\infty}, \frac{\mathbb{Q}_p}{\mathbb{Z}_p}) = 0$ is clear, since the first in connected and the second constant. For the second we can't give a complete proof but we make some observation. If we take the same statement with changing p with some prime l different from the characteristic of the field, then the statement $Hom(B[l^\infty], \frac{\mathbb{Q}_l}{\mathbb{Z}_l}) = 0$ is an easy consequence of A.4.8. Indeed, everything is étale and so we can pass to the associated Γ_k modules and we have to show that $Hom(T_l(B), \mathbb{Z}_l) = 0$. But, thanks to A.4.8, the eigenvalues of the frobenius acting on $T_l(B)$ are different from 1, while on the second all the eigenvalues are 1. This implies that there are no equivariant maps between them. When $l = p$, exactly the same reasoning works changing the notion of Γ_k modules with the notion of Dieudonné modules and étale cohomology with crystalline cohomology.. □

5.3 Conclusion of the proof: Faltings height is an height

In this section we will give a sketch of the proof of the following theorem:

Theorem 5.3.1. *For every integer $g \geq 1$ and even integer $r \geq 2$ there exists a constant $C(g,r)$, that depends only on g and r such that the following holds.*
Let A an abelian variety of dimension g defined over $\overline{\mathbb{Q}}$ equipped with a principal polarization defined by some symmetric ample line bundle L. Then

$$|d(A, L^{r^2}) - \frac{1}{2}h_F(A)| \leq C(g,r)log((max(d(A, L^{r^2}), 1) + 2)$$

This is enough to prove the Tate conjecture. Indeed we take an abelian variety over a number field K and to prove the conjecture we can enlarge our ground field with a finite extension and replace A with an isogenous abelian variety, so that we can assume, thanks to 3.2.4 and 4.3.1, that A is principally polarized by a symmetric line bundle and with semistable reduction. Recall the setting of 1.2.3 3). Fix a sub l divisible group of $G = \{G_n\}$ of $A[l^\infty]$ over k such that $B_n := \{\frac{A}{G_n}\}$ are all principal polarized by a symmetric line bundle. We have to show that the B_n fall into finitely many isomorphism classes. Then, thanks to A.1.36, all the B_n have semistable reduction. Using A.1.16 we find a finite extension F such that a delta structure is defined over it. Using 5.1.1 and 5.3.1 they have bounded delta height so they fall into finitely many classes as polarized abelian variety over $\overline{K}$, using A.3.2. So by 3.3.4 they are fall into finitely many isomorphism classes, as polarized abelian variety over F. Now just observe that, as in the proof of 3.3.4, using A.1.17 and 3.3.3, this implies that they fall into finitely many classes over K.

5.3.1 Comparison of heights

There are several ways to prove the comparison theorem or some of its possible variants. The common point of all of them is the use of Arakelov geometry. The original approach of Faltings pass trough a compactification $\mathcal{A}_g$ of A_g, a coarse moduli space for principally polarized abelian varieties of dimension g, and some computations in the boundary of A_g inside $\mathcal{A}_g$. We will follow a different pattern, more elementary, following [Paz12]. We start giving the necessary definition from Arakelov geometry.

Arakelov Geometry

Definition 5.3.2. A metrized vector bundle on $\mathcal{O}_K$ is a pair $(L, (|-|_\sigma)_{\sigma:K\to\mathbb{C}})$, where L is a vector bundle over $\mathcal{O}_K$ and $|-|_\sigma$ is a norm on $L \otimes_\sigma \mathbb{C}$.

Definition 5.3.3. If L is a metrized line bundle we define it's degree as

$$deg(L, (|-|_\sigma)_{\sigma:K\to\mathbb{C}}) = log(\left|\frac{L}{s\mathcal{O}_K}\right|) - \sum_\sigma log||s||_\sigma$$

where s is any non zero section of L.
If $(L, (|-|_\sigma))$ is any metrized vector bundle we define

$$Deg((L, (|-|_\sigma))) = Deg(Det(L, (|-|_\sigma)))$$

Remark. As in 5.1, one can show that the degree does not depend on the choice of s.

Definition 5.3.4. If X is a projective variety over K. A metrized vector bundle on X is a pair $(L, (|-|_{p,\sigma})_{x\in X(K_\sigma),\sigma:K\to\mathbb{C}})$ where L is a vector bundle over X and $|-|_{x,\sigma}$ is a norm on each fiber $L_x \otimes_\sigma \mathbb{C}$ for every $x \in X(K_\sigma)$ that satisfies the following continuity condition: For every open subset U of X and every $s \in H^0(U, L)$ the map

$$U(K_\sigma) \to [0, +\infty)$$
$$x \mapsto |f_x|_{x,\sigma}$$

is continuous

Remark. Metrized vector bundle are stable for all the usual operation on vector bundle, like finite direct sums, tensor products, determinants, duals and pullback.

Definition 5.3.5. If $x \in X(\mathcal{O}_K)$ and L is a metrized vector bundle, we define the degree of L at x as $Degx^*L$ and the height of L at x has $h_L(x) := \frac{1}{[K:\mathbb{Q}]}Degx^*L$

Example. The Faltings height of an Abelian variety A, defined in 5.1.2, can be seen as the degree of a metrized line bundle. Indeed if $\pi : \mathcal{A} \to \mathcal{O}_K$ is the Neron model of A with unit section ϵ , we can make $\pi_* \Omega^g_{\mathcal{A}/\mathcal{O}_K}$ a metrized line bundle observing that

$$(\pi_* \Omega^g_{\mathcal{A}/\mathcal{O}_K}) \otimes_\sigma \mathbb{C} = H^0(\mathcal{A}_\sigma(\mathbb{C}), \Omega^g_{A_\sigma})$$

and defining a norm on this space by

$$|\alpha|_\sigma = \int_{\mathcal{A}_\sigma(\mathbb{C})} \alpha \wedge \overline{\alpha}$$

Then we get $h_F(A) = \frac{1}{[K:\mathbb{Q}]} Deg(\pi_* \Omega^g_{A/\mathcal{O}_K}, (|-|_\sigma))$

Example. Also the usual height on the projective space can be recovered as the degree of a metrized line bundle of $\mathbb{P}^n$. Indeed we can make $\mathcal{O}(1)$ a metrized line bundle setting for every $f \in H^0(\mathbb{P}^n_{K_\sigma}, \mathcal{O}(1))$ and every $P \in \mathbb{P}^n(K_\sigma)$:

$$|f_P|_{\sigma,P} = min_{0 \le i \le n, x_i(p) \neq 0}(|\frac{f}{x_i}(P)|_\sigma)$$

where the x_i are the canonical generators of $\mathcal{O}_{\mathbb{P}^n}(1)$. Then one has $h(P) = \frac{1}{[K:\mathbb{Q}]} Deg P^* L$ for every $P \in \mathbb{P}^n(\mathcal{O}_K)$.

Finally we recall an important invariant associated to an ample line bundle L over A. The choice of a basis of global sections induces a map $f : A \to \mathbb{P}^n_K$ and an isomorphism $f^* \mathcal{O}_{\mathbb{P}^n}(1) \simeq L$. With this isomorphism L becomes a metrized line bundle and hence give as a function $h_L : A(K) \to \mathbb{R}$.

Theorem 5.3.6. *There is a unique quadratic function*

$$\tilde{h}_L : A(K) \to \mathbb{R}$$

such that $\tilde{h}_L = h_L + O(1)$ and $\tilde{h}_L(0) = 0$ and does not depend on the choice of the basis.

Proof. [BG06] Theorem 9.2.8 □

Definition 5.3.7. The Neron Tate height associated to L is the unique function $\tilde{h}_L : A(K) \to \mathbb{R}$ in the previous theorem.

Comparison

Let A be a g-dimensional principally polarized abelian variety defined over K whose polarization is induced by a symmetric ample line bundle. Let r be an even positive integer. We will assume that all the r^2 torsion points of A are rational enlarging our base field if needed. Observing that $K(L^{r^2}) = A[r^2]$, choose a delta structure over K and a rigidification of L at the origin, i.e. an isomorphism between the fiber of L in 0 and K. We note that a delta structure determines a family of isomorphisms $i_x : t_x^* L^{r^2} \to L^{r^2}$ for $x \in A[r^2]$, and the choice of the rigidification determines an isomorphism $j : [r]^* L \to L^{r^2}$. For any $x \in A[r^2]$ define

$$\psi_x : H^0(A, L) \to H^0(A, L^{r^2})$$
$$s \mapsto i_x \circ t_x^* \circ j \circ [r]^*(s)$$

and

$$\psi : \oplus_{x \in \frac{A[r^2]}{A[r]}} H^0(A, L) \to H^0(A, L^{r^2})$$
$$s \mapsto \sum_{x \in \frac{A[r^2]}{A[r]}} \psi_x(s)$$

an observe that is a non zero map, such that the image is an equivariant subspace of $H^0(A, L^{r^2})$ under the action of $G(L^{r^2})$. By 4.1.4 and counting dimension this map is an isomorphism. We will study the height of A with respect to a delta structure on L^{r^2} and we will denote it with $d(A, L^{r^2})$.

The key input comes from the existence of the so called M.B. model $(\mathcal{B}, \mathcal{L}^{r^2}, (\epsilon_x)_{x \in A[r^2]})$ over a finite extension N of K, of A, where $\mathcal{B}$ is a quasi projective group scheme $\pi : \mathcal{B} \to Spec(\mathcal{O}_K)$ over $\mathcal{O}_K$, $\mathcal{L}^{r^2}$ is a metrized line bundle over $\mathcal{B}$ and, for every $x \in A[r^2]$, ϵ_x is a section $Spec(\mathcal{O}_N) \to \mathcal{B}$. The tree main properties of $\mathcal{B}$ that will allow us to get the comparison are the following. First of all, the Faltings height of A is related to the degree of the metrized vector bundle $\pi_*\mathcal{L}^{r^2}$, with the following equality:

$$\frac{deg\pi_*\mathcal{L}^{r^2}}{[N:\mathbb{Q}]} = -\frac{1}{2}h_F(A) - \frac{g}{4}log(2\pi)$$

Secondly, there exists an isomorphism $i : A_{\overline{K}} \to \mathcal{B}_{\overline{K}}$ such that the geometric point associated to ϵ_x corresponds to $x \in A[r^2](\overline{K})$, for every $x \in A[r^2]$.
Finally, the degree of the line bundle is related to the Neron-Tate height $\tilde{h}_L$ in the following way:

$$\frac{Deg(\epsilon_x^*\mathcal{L}^{r^2})}{[N:\mathbb{Q}]} = \tilde{h}_L(x)$$

For details, see [Paz12] Definition 3.1 and Theorem 3.4. With this B.M. model and the language of Arakelov geometry, one can compare the different heights. In particular, we observe that for $x = 0$, $\frac{Deg(\epsilon_x^*\mathcal{L}^{r^2})}{[N:\mathbb{Q}]} = 0$. Denote with $\mathcal{F}$ the metrized line bundle $\pi_*\mathcal{L}^{2g}$ over $\mathcal{O}_N$. The construction of Pazuki, is based on the fact that, using the existence of a lifting of the isomorphism $\psi : \oplus_{x\in\frac{A[r^2]}{A[r]}} H^0(A, L) \to H^0(A, L^{r^2})$ to an injection $\mathcal{F} \subseteq \pi_*\mathcal{L}^{r^2}$, one can construct a map $i_N : \mathcal{B}_N \to \mathbb{P}_N^{r^{2g}-1}$ with the following two proprieties:
1)When base changed to the algebraically closure (an composed with the isomorphism i) is the delta embedding.
2)The heights satisfies the following equality:

$$h(i_N(x)) = h_{\mathcal{F}}(i_N(x)) - \frac{1}{[N:\mathbb{Q}]}Deg\pi_*\mathcal{L}^{r^{2g}}$$

where $h_{\mathcal{F}}$ is the height on $\mathbb{P}(\mathcal{F}_N) \simeq \mathbb{P}_N^{r^{2g}-1}$ attached to the metrized line bundle associated to $\mathcal{O}_{\mathcal{F}}(1)$
For details see [DDSMS99] pages 14-16-17, we just observe that the last equality can be deduced from an isomorphism between two metrized line bundles, $\mathcal{O}_{\mathcal{F}}(1) \simeq \pi^*\pi_*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^{r^{2g}-1}}(1)$, and this partially explain the power of a uniform language to deal with different heights.
By the properties of $B.M.$ model explained before, we get for every x,

$$h(i_N(x)) = h_{\mathcal{F}}(i_N(x)) + \frac{1}{2}h_F(A) + \frac{g}{4}log(2\pi)$$

In particular, by the property 1 and the above formula applied to $x = 0$, we get that

$$d(A, L^{r^2}) = h_{\mathcal{F}}(i_N(0)) + \frac{1}{2}h_F(A) + \frac{g}{4}log(2\pi)$$

and so that to obtain the comparison one has just to bound $h_{\mathcal{F}}(i_N(0))$. To deal with this, the idea is to use the inclusion $\mathcal{F} \subseteq \pi_*\mathcal{L}^{r^2}$ to compare the degree of the two vector bundles and then use the fact that $\frac{Deg(\epsilon_x^*\mathcal{L}^{r^2})}{[N:\mathbb{Q}]} = \tilde{h}_L(x)$ and that this is zero when applied to $x = 0$. The inclusion $\mathcal{F} \subseteq \pi_*\mathcal{L}^{r^2}$ induces, by adjuction, a map $\pi^*\mathcal{F} \to \mathcal{L}^{r^2}$. One can show that the image of this map is in the form $I_{B_{\mathcal{F}}}\mathcal{L}^{r^2}$ for some sheaf of ideals $I_{B_{\mathcal{F}}}$ over $\mathcal{B}$ and we denote $B_{\mathcal{F}}$ the closed subscheme defined by $I_{B_{\mathcal{F}}}$. Using [Paz12] Theorem 3.4, we get that $B_{\mathcal{F}}$ has empty generic fiber and hence that $\epsilon^*B_{\mathcal{F}}$ is a divisor that we will write $\epsilon^*B_F = \sum_v \beta_v(\mathcal{L}^{r^2}, \mathcal{F})\mathfrak{p}_v$. We will also consider their Archimedean counterparts, defined as

$$\beta_\sigma(\mathcal{L}^{r^2}, \mathcal{F}) := \frac{1}{2}log(\sum_{1\leq i\leq n} |u_i|_\sigma^2(0))$$

where u_i is any orthonormal basis of $\mathcal{F}_\sigma$, a subspace of $H^0(\mathcal{B}_\sigma, \mathcal{L}_\sigma^{r^2})$. Then one can show the following:

Lemma 5.3.8.

$$h_{\mathcal{F}}(i_N(0)) = -\frac{1}{[N:\mathbb{Q}]}\Big(\sum_v \beta_v(\mathcal{L}^{r^2}, \mathcal{F})log(N(\mathfrak{p}_v)) + \sum_\sigma \beta_\sigma(\mathcal{L}^{r^2}, \mathcal{F})\Big)$$

Proof. See [Paz12] Proposition 4.1. ∎

So we found:

$$d(A, L^{r^2}) - \frac{1}{2}h_F(A) = -\frac{1}{[N:\mathbb{Q}]}\Big(\sum_v \beta_v(\mathcal{L}^{r^2}, \mathcal{F})log(N(\mathfrak{p}_v)) + \sum_\sigma \beta_\sigma(\mathcal{L}^{r^2}, \mathcal{F})\Big) + \frac{g}{4}log(2\pi)$$

The next two propositions conclude the proof.

Lemma 5.3.9.

$$\frac{1}{[N:\mathbb{Q}]}\Big(\sum_v \beta_v(L^{r^2}, \mathcal{F})log(N(\mathfrak{p}_v))\Big) \leq \frac{g}{2}r^{2g}log(r)$$

Proof. See [Paz12] Lemma 5.3. □

Lemma 5.3.10.

$$\Big|\frac{1}{[N:\mathbb{Q}]}\sum_\sigma \beta_\sigma(\mathcal{L}^{r^2}, \mathcal{F})\Big| \leq C(g,r)log((max(d(A, L^{r^2}), 1) + 2)$$

for some positive constant $C_1(g, r)$.

Proof. See [Paz12] Lemma 5.4, Proposition 5.5, Remark 1.2. □

Appendix A

Collection of facts

A.1 Generalities on Tate modules and abelian varieties

A.1.1 General theorems

Proposition A.1.1. *Every abelian variety of dimension g over the complex number is isomorphic to $\frac{\mathbb{C}^g}{\Lambda}$ for some lattice Λ of rank g. Every isogeny between abelian variety $f : \frac{\mathbb{C}^g}{\Lambda_A} \to \frac{\mathbb{C}^g}{\Lambda_B}$ is induced by an inclusion $\Lambda_A \to \Lambda_B$ with finite cokernel of $deg(f)$.*

Proof. See [SC86] Chapter 4.1 $\square$

Proposition A.1.2. *Every abelian variety over a field is isogenous to a product of powers of pairwise not isogenous simple abelian varieties.*

Proof. See [MVdG13] Corollary 12.5. $\square$

Proposition A.1.3. *For every abelian varieties A of dimension g over a field k, the map $Deg : End(A) \otimes \mathbb{Q} \to \mathbb{Q}$ is a homogeneous polynomial function of degree $2g$.*

Proof. See [MVdG13] Proposition 12.15 $\square$

Proposition A.1.4. *For every abelian varieties A over a field k and every $\alpha \in End(A)$ there exists a unique monic polynomial $P_\alpha \in \mathbb{Z}[x]$ of degree $2g$ such that $P_\alpha(r) = deg(\alpha - r)$ for all $r \in \mathbb{Z}$. The same is true if we change $End(A)$ with $End(A) \otimes \mathbb{Q}$ and $\mathbb{Z}$ with $\mathbb{Q}$.*
Moreover P_α is the characteristic polynomial of $V_l(\alpha)$ acting on $V_l(A)$ when $l \neq char(k)$.

Proof. See [Mil08] Theorem 10.9, Proposition 10.13 and Proposition 10.20. $\square$

Definition A.1.5. $Pic^0(A) : Var_k \to Ab$ is defined by the rule

$$Pic^0(A)(T) = \left\{ \begin{array}{c} L \in Pic(A \times T) \text{ such that } L_{|A\times\{t\}} \text{ is invariant by translation} \\ \text{and } L_{|\{0\}\times A} \text{ is trivial} \end{array} \right\}$$

Proposition A.1.6. *1)The functor $Pic^0(A)$ is representable by an abelian variety A'.*
2) $Pic^0(A')(T) = \frac{Pic(A\times T)}{\pi_T^*(Pic(T))}$
3)$A'' = A$

Proof. See [MVdG13] Chapter 6 and 7. $\square$

Definition A.1.7. The Poincaré bundle $\mathcal{P}_A$ of A is the universal line bundle on $A \times A'$

Proposition A.1.8. *$(\mathcal{P}_A)_{\{t\}\times A'}$ is trivial if and only if $t = 0$.*
It satisfies $(m, n)^*\mathcal{P}_A = \mathcal{P}_A^{mn}$.
A line bundle L over A defines a morphism $\psi_L : A \to A'$ such that $(\psi_L \times id)^*\mathcal{P}_A = m^*L \otimes p^*L^{-1} \otimes q^*L^{-1}$.
$K(L)$ is the kernel of this morphism and it is the set of point of A such that $t_x^*L = L$.
If $L \in Pic^0(A)$, $[n]^*L = L^n$, $\psi_L = 0$.
If L is ample ψ_L is an isogeny.

Proof. See [MVdG13] Chapter 6 and 7. $\square$

Proposition A.1.9. *$[n] : A \to A$ is an isogeny and it is étale if and only if $char(k) \not| n$.*

Proof. See [Mil08] Theorem 7.2. $\square$

Proposition A.1.10. *If $f : A \to B$ is an isogeny we have a perfect pairing $e_f : ker(f) \times ker(f') \to \mathbb{G}_m$. This pairing satisfies the usual property of the pairing.*

Proof. We will prove it in chapter 2 section 1. For more details see [MVdG13] Chapter 7. $\square$

Remark. Observe that if f is a polarization then the pairing is in the following form:

$$ker(f) \times ker(f) \to \mathbb{G}_m$$

.

Definition A.1.11. We will denote with e_A the perfect pairing associated to $f = [n] : A \to A$, it is a perfect paring $A[n] \times A'[n] \to \mathbb{G}_m$. It is called the Weil pairing.
If λ is a polarization we get a paring $A[n] \times A[n] \to \mu_n$ composing the previous paring with the map $id \times \lambda$. It is called the Weil pairing associated to λ and it is denoted with e_A^λ

Definition A.1.12. A line bundle is said non degenerate if $K(L)$ is finite.
An isogeny $f : A \to A'$ is said a polarization if, in a finite extension K, is in the form ψ_L for some $L \in Pic(A_K)$. We say that f in principal if it is an isomorphism.

Proposition A.1.13. *If $char(k) \not| n$ then $A[n](\overline{k}) \simeq (\frac{\mathbb{Z}}{n\mathbb{Z}})^{2g}$.*

Proof. This follows from 2.0.1 and the fact that if L is an ample symmetric line bundle $[n]^*L = L^{n^2}$. $\square$

Proposition A.1.14. *If A is an abelian variety over a noetherian henselian local domain S, then the functor $End(A|S)$ is representable by a finite unramified group scheme.*

Proof. See [MVdG13] Proposition 7.14 for fields, the same proof works over henselian local domain. $\square$

Proposition A.1.15. *If $p : G \to S$ is a group scheme with unit section ϵ, then $\Omega^1_{G/S} \simeq p^*e^*\Omega^1_{G/S}$. If G is smooth and S is a local ring $\Omega^1_{G/S}$ is free of dimension G*

Proof. See [BWR90] Proposition 2 Pag. 102 $\square$

Proposition A.1.16. *[Zar74] Let A be an Abelian variety over a field K, and let m be a natural number not divisible by the characteristic of the field. There exists a finite separable extension L such that if B is an abelian varieties isogenous to A, then all the m torsion points of B are L rationals.*

Proof. It is enough to deal with $d = l^n$ for some prime l different with the characteristic of the field.
Then consider $G = Im(\rho)$ where $\rho : \Gamma_K \to GL_{2g}(T_l(A))$. Up to replace k with a finite extension we can assume that $G \subseteq 1 + lM_l(\mathbb{Z}_l)$. Then it is a pro-$l$ compact l-adic Lie group thanks to Cartan's Theorem. Observe that $K(B[l^n]) \subseteq K(A[l^\infty])$ and its degree over K is bounded by some constant C that depends only on g, l, n. Consider the intersection of the kernels of all the maps from G to finite groups with order less or equal to C. By [DDSMS99], Corollary 1.21, it is an open subgroup U. The extension associated to $\rho^{-1}(U)$ is the required extension. $\square$

Proposition A.1.17. *Let (A, L) be a polarized abelian variety over a field k and K a Galois extension of k. The the set of isomorphism classes of polarized abelian varieties over k that are isomorphic to A over K is in bijection with $H^1(Gal(K|k), Aut(A_K, L_K))$.*

Proof. See [Ser97a] Proposition 5, Page 131 for the proof for general quasi projective varieties. The proof show that everything works in same way adding the dependence from a polarization. $\square$

A.1.2 Tate module, Neron model and good reduction

Definition A.1.18. We say that a sequence of group scheme $0 \to H \to G \to F \to 0$ is exact if the first map is a closed immersion and the second identifies F with the categorical quotient of G and H.

We recall the following theorems.

Theorem A.1.19. *1)There is an equivalence of categories between étale finite group scheme over a field k and Γ_k discrete modules.*
2)If k is of characteristic zero, then every finite group scheme is étale .
3) There exists two exact endofunctor $(-)^0$ and $(-)^{ét}$ in the categories of group scheme over k such that every finite group scheme G fits in an exact sequence

$$0 \to G^0 \to G \to G^{ét} \to 0$$

and $G^{ét}$ is étale and G^0 is connected.
4)The last point is true if we change k with a complete noetherian local ring and we assume G flat.

Proof. 1) is [Pin04],Theorem 12.2, 2) is [Pin04], Theorem 13.2, 3 is [Pin04] Proposition 15.3, 4 is [Ntls10] L06 Theorem 16. □

Definition A.1.20. Let R be a ring. A p divisible group over R of height h is a collection $\{G_n, i_n : G_n \to G_{n+1}\}_{n\in\mathbb{N}}$ such that G_n is a finite flat group scheme of rank p^{nh} and the sequence

$$0 \to G_n \overset{i_n}{\to} G_{n+1} \to \overset{p}{G}_{n+1}$$

is exact

Theorem A.1.21. *1) There exists an equivalence of categories étale p-divisible group over a field k and Γ_k discrete $\mathbb{Z}_p$ modules.*
2)If we consider $\varinjlim_i G_i$ as a fppf sheaf, then the multiplication by p is surjective and this give us an embedding of the category of p divisible in the category of fppf sheaves.
3)There exists two exact endofunctor $(-)^0$ and $(-)^{ét}$ in the categories of p divisible group over k such that every p divisible group G fits in an exact sequence

$$0 \to G^0 \to G \to G^{ét} \to 0$$

and $G^{ét}$ is étale and G^0 is connected
4)The previous point remains true if we change k with a complete local ring.
5)The multiplication by p^m induces an exact sequence

$$0 \to G_v \overset{i}{\to} G_{v+w} \overset{p^v}{\to} G_w \to 0$$

where i is the composition of the necessary i_j.

Proof. See [SC86] Chapter 3.6 □

Example. 1)$\mathbb{G}_m(p)_v := \mu_{p^v}$
2)$A[p^\infty]_v := A[p^v]$, if A is an abelian variety.
3)$\frac{\mathbb{Q}_l}{\mathbb{Z}_l}_v = \frac{\mathbb{Z}}{l\mathbb{Z}}$

Definition A.1.22. If G is p-divisible group over a domain with fraction field G, we define

$$T_p(G) := \varprojlim_v G_v(\overline{K}) \qquad \Phi(G) := \varinjlim G_v(\overline{K})$$

where the limit is taken first with respect to the projection and then to the inclusion. Moreover we define $V_p(G) = T_p(A) \otimes \mathbb{Q}_p$
If A is an abelian variety we define $T_p(A) := T_p(A[p^\infty])$ and $V_p(A)$ in the same way.

Example. If A is an abelian variety of dimension g and l is different from the characteristic of the field, then $T_l(A)$ is a free module $\mathbb{Z}_l$ of rank $2g$.

Proposition A.1.23. *If G is a p-divisible group or a finite group of order a power of p over a field of characteristic p we have a morphism induced by the Frobenius $F : G^{(p)} \to G$ and a morphism induced by the dual of Frobenius $V : G \to G^{(p)}$ such that $V \circ F = [p] = F \circ V$.*
If G is p divisible then V and F are epimorphisms in category of fppf sheaves.

Proof. See [Pin04] Chapter 14 and 15 for the statements about group schemes. The proofs are similar for p-divisible groups. The last statement follows from [MVdG13] 10.13. and [Pin04] Proposition 15.6 □

Definition A.1.24. Let R be a noetherian complete local ring with residue field k of characteristic $p > 0$. Let $\mathcal{B} = R[[x_1, ..., x_n]]$ and $e : \mathcal{B} \to R$ the map that send each x_i in 0. A formal group over R of dimension n is a map $\mathcal{B} \to \mathcal{B}\tilde{\otimes}_R\mathcal{B}$, where the latter is the completed tensor product, such that m is coassociative and cocommutative with e as a counit.
A formal group is said divisible if the multiplication by p is finite free. We will denote with I the ideal $(x_1, .., x_n)$.

Theorem A.1.25. *The functor that send a divisible formal group $\mathcal{B}$ to the connected p-divisible group $\{\frac{\mathcal{B}}{([p^v]^*\mathcal{A})}\}_{v\in\mathbb{N}}$ gives an equivalence of categories between the category of divisible formal groups and the category of connected p-divisible group.*

Proof. See [Ntls10] L9, Page 10. □

Definition A.1.26. If G is p-divisible group over a complete noetherian domain we define the tangent space to G, t_G, as the tangent space of the formal group $\mathcal{B}$ associated to the generic fiber of G^0 and $Dim(G) = Dim(t_G)$

Definition A.1.27. Let R be a domain with fraction field K and A an abelian variety over K. A Neron model $\mathcal{A}$ for A is a smooth, commutative, separated and quasi projective group scheme with generic fiber A and such that $Hom(X, \mathcal{A}) = Hom(X_K, A)$ for every smooth separated scheme X over R.

Observe that if it exists it is unique up to a canonical isomorphism.

Theorem A.1.28. *If R is a Dedekind domain, with fraction field K and A is an abelian variety over K there exist the Neron model of A over R.*

Proof. See [BWR90] Theorem 3 Pag 19. □

Definition A.1.29. Let R be a domain. A commutative smooth connected separated quasi projective group scheme G over R is a semiabelian variety if there exists an exact sequence of group scheme

$$0 \to T \to G \to A \to 0$$

where T is a torus and A is an abelian variety.

Definition A.1.30. Let R be a domain with fraction field K and A an abelian variety over K. We say that A has good reduction at same place v of R if the base change of the connected component of the Neron model of A at that place is an abelian variety. We say that A has semistable reduction at same place v of R if the base change of the connected component Neron model of A at that place is a semiabelian variety.

Proposition A.1.31. *Let R be a domain with fraction field K and A an abelian variety over K. Then it has good reduction outside finitely many places.*

Proof. See [BWR90] Theorem 3, Page 19. □

Theorem A.1.32. *Let R be a complete discrete valuation ring with fraction field K and A an abelian variety over K. Then there exists a finite extension L of K such that A acquires semistable reduction.*

Proof. See [BWR90] Theorem 1 Pag. 181. □

Corollary A.1.33. *Let R be a Dedekind domain with fraction field K and A an abelian variety over K. Then there exists a finite extension L of K such that A acquires semistable reduction at every place.*

Corollary A.1.34. *Let R be a Dedekind domain with fraction field K and A an abelian variety over K with semistable reduction. Then the connected component of the Neron model is stable by base change*

Proof. See [BWR90] Corollary 4 Pag 183. □

Proposition A.1.35. *Let R be a complete discrete valuation ring with fraction field K and A an abelian variety over K. Then A has good reduction if and only if the representation on $T_l(A)$ is unramified.*

Proof. See [ST68] Theorem 1 □

Proposition A.1.36 (Unipotent)**.** *Let R be a complete discrete valuation ring with fraction field K and A an abelian variety over K. Then A has semistable reduction if and only if the action of the inertia on $T_l(A)$ is unipotent.*

Proof. See [BWR90] Theorem 5 Pag 183.

□

A.2 Some non commutative algebra

Theorem A.2.1. *Let R be a k algebra and $E = End_k(V)$ for some faithful semisimple R module V. Then $Centr_E(Centr_E(R)) = R$*

Proof. See [Jac09] Thm. 4.10 □

Theorem A.2.2. *In a semisimple algebra every right ideal is generated by an idempotent*

Proof. This is a classical result. The nicer proof is in [Ntls10] L20, Proposition 4.4. □

Theorem A.2.3. *If A is semisimple algebra over a field of k of characteristic zero, then $A \otimes_k k'$ is semisimple for every field extension $k \to k'$.*

Proof. See [Ntls10] Proposition 5.2. □

Theorem A.2.4. *Let R be any Dedekind domain whose quotient field is a global field. Then for each R order Λ in a semisimple K algebra A, and for each positive integer t, there are only finitely many isomorphism classes of right Λ lattices of R rank at most t*

Proof. See [Rei75] Theorem 26.4 □

Theorem A.2.5. *Let A be a semisimple K algebra, where K is the quotient field of a noetherian integrally closed domain R of characteristic zero.*
1) Every R order is contained in a maximal R order in A. There exists at least one maximal R order in A.
2)Let Λ be a maximal order in A. Then every right Λ lattice is projective.

Proof. See [Rei75] Corollary 10.4 and Corollary 21. 5 □

A.3 Some algebraic geometry

Definition A.3.1. Let K be a global field, i.e a finite extension of Q, where Q is $\mathbb{F}_p(T)$ or $\mathbb{Q}\}$. Let Ω_F denotes the set of places of F, where F is any finite extension of K and $x = (x_0 : ... : x_n) \in \mathbb{P}^n(\overline{K})$. We define the height of x as

$$h(x) = \frac{1}{[F:Q]} \sum_{v \in \Omega_F} log(max_{0 \leq i \leq n}(|x_i|_v))$$

where F is any finite extension containing $K(x_1, ..., x_n)$.

Remark. The height is well defined thanks to the product formula and the behavior of valuations over finite extension.

Proposition A.3.2. *With the notation of the previous definition, for everyn, $m \in \mathbb{N}$ there exists only finitely many point in $\mathbb{P}^n(\overline{K})$ defined over an extension of degree smaller then n with heights smaller then m.*

Proof. See [SC86] Chapter 6.2 and [BG06] Example 9.4.20. □

Theorem A.3.3. *[Grothendieck duality] If $f : X \to Y$ is a smooth morphism of dimension g between noetherian scheme and $F \in D(X)$,$G \in D(Y)$ then there is a natural isomorphism*

$$Hom_{D(X)}(F, (f^*(G) \otimes \omega_{X/Y})[g]) \simeq Hom_{D(Y)}(Rf_*F, G)$$

.

Proof. See [Har63] □

Theorem A.3.4. *Let $\pi : X \to S$ be a proper morphism of noetherian scheme and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ module which is flat over $\mathcal{O}_S$. Then:*

1. *If for some integer i there is some integer d such that for all $s \in S$ we have $dim_{k(s)} H^i(X_s, \mathcal{F}_s) = d$, then $R^i\pi_*\mathcal{F}$ is locally free of rank d and $(R^{i-1}\pi_*\mathcal{F})_s \simeq H^{i-1}(X_s, \mathcal{F}_s)$ for every $s \in S$*

2. *If for some integer i and some $s \in S$ the map*

$$(R^i\pi_*\mathcal{F})_s \to H^i(X_s\mathcal{F}_s)$$

is surjective then

$$(R^{i-1}\pi_*\mathcal{F})_s \to H^{i-1}(X_s, \mathcal{F}_s)$$

is surjective if and only if $R^i\pi_*\mathcal{F}$ is locally free in a neighborhood of s.

Proof. See [Har77] Section 12 and [Fa05] Theorem 5.12. □

Theorem A.3.5. *Suppose that $X \to Y$ is a proper math between smooth variety. Then for every line bundle L over X we have $\chi(L) = Deg(ch_1(L)Td(X))_{(n)}$.*

Proof. See [Ari10] Corollary 3.8 for the proof and the notation. □

A.4 Some theory of complete local rings

Proposition A.4.1. *Let G be a connected p-adic analytic group over some p-adic field K. Then for every extension L on K and every $x \in G(L)$, $\lim_{n \to +\infty} p^n x = 0$.*

Proof. See [Lie00]

□

Proposition A.4.2. *Let G be a connected p-adic analytic group over some p-adic field K. Then for every extension L on K there exists a functorial analytic homomorphism $G(L) \to t_G(L)$ and it is an isomorphism in a neighborhood of the identity.*

Proof. See [Lie00] □

Proposition A.4.3. *If K is a number field there exists a finite extension such that every ideal become principal.*

Proof. The integral closure $\overline{\mathbb{Z}}$ of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$ is a Bézout domain. The class group of K is finite. If $I \in Pic(\mathcal{O}_K)$ then I becomes principal in $\overline{\mathbb{Z}}$. The generator is defined in a finite extension. So I becomes principal in a finite extension. Now just repeat the process for every class in the class group. □

Proposition A.4.4. *Suppose that R is a complete noetherian local ring and G a quasi finite scheme over it. Then there exists a unique decomposition $G = G^f \coprod G^\eta$, where G^f is finite and G^η has empty special fiber. The formation of the finite part is functorial and preserve products so that it sends group scheme in group scheme.*

Proof. This follows from the Zariski main theorem and the fact that every finite group scheme of a complete noetherian ring is disjoint union of local scheme. □

Definition A.4.5. Let K be a p-adic field and denote with $\mathbb{C}_K$ the completion of the algebraic closure of K. A representation V of Γ_K is Hodge Tate if $V \otimes \mathbb{C}_K \simeq \oplus_i \mathbb{C}_K(i)^{n_i}$. We say that a one dimensional Hodge-Tate representation is of weight m if $V \otimes \mathbb{C}_K \simeq \mathbb{C}_K(m)$

Proposition A.4.6. *Let K be a p-adic field, denote with $\mathbb{C}_K$ the completion of the algebraic closure of K and let V be a Γ_K representation. Then:*
1)$H^0(\mathbb{C}_K, \Gamma_K) = K$.
2)$H^0(\mathbb{C}_K(i), \Gamma_K) = 0$ if $i \neq 0$.
3)$H^1(\mathbb{C}_K(i), \Gamma_K) = 0$ if $i \neq 0$.
4)The Hodge-Tate weight of a one dimensional representation is well defined.
5)If V is unramified, then it is Hodge-Tate of weight 0.
6)If V is one dimensional and Hodge-Tate of weight 0, then the image of the inertia is finite.
7)$H^1(\mathbb{C}_K(i), \Gamma_K)$ is in bijection with the set of isomorphism classes of continuous exact sequences $0 \to \mathbb{C}_K(i) \to W \to \mathbb{C}_K \to 0$

Proof. All of this can be found in [CB09] □

Proposition A.4.7. *Let G be a group, H a subgroup of finite index, K any subgroup and $S = K\backslash M/H$. Then for every representation V of H we have that*

$$(Ind_G^H(V))_{|K} = \oplus_{s\in S} Ind_K^{H_s}(W_s)$$

where $H_s = sHs^{-1} \cap K$ and W_s is the representation of H_s given by $g * x = s^{-1}gs * x$

Proof. For the statement about finite group see [Ser77] Chapter 7,3 Prop. 22.. The proof in this situation is analogous. □

Proposition A.4.8. *Suppose that X is a smooth proper scheme over a finite field K of cardinality q. Then the eigenvalues of the Frobenius acting on $H^i_{ét}(A, \mathbb{Z}_l)$ have complex absolute value $|q|^{\frac{i}{2}}$.*

Proof. See [Del80]. □

Appendix B

A proof of 4.3.2 without identifications

Theorem B.0.1. *The map $End(A) \to T_l(A)$ is surjective.*

Proof. Recall the following commutative diagram:

$$\begin{tikzcd}
k_H & \frac{H}{\mathfrak{p}_S} \arrow[l, "\pi_{\frac{H}{\mathfrak{p}_S}}"'] \arrow[d, "r_S"] & H \arrow[l, "\pi^S_H"'] \arrow[ll, bend right, "\pi_H"'] \arrow[r, "i_H"] \arrow[d, "j_S"] & \overline{K} \\
 & k_S & S \arrow[l, "\pi_S"'] \arrow[ur, "i_S"'] &
\end{tikzcd}$$

- Step 1. We have the following chain of inclusions:

$$ET_l(r_S)ET_l(\pi^S_H)ET_l(i_H)^{-1}(End_G(T_l(A_{\overline{K}}))) \overset{(1)}{=} ET_l(\pi_S)ET_l(i_S)^{-1}(End_G(T_l(A_{\overline{k}})))$$
$$\overset{(2)}{\subseteq} End_{G_S}(T_l(A_{k_S})) \overset{(3)}{\subseteq} End(A_{k_S}) \otimes \mathbb{Z}_l \overset{(4)}{=} EA(r_S)(End(A_{\frac{H}{\mathfrak{p}_S}}) \otimes \mathbb{Z}_l)$$

.
1) Is clear by the commutativity of the diagram.
2) Is clear since the action of G_S on $T_l(\overline{k})$ is compatible with the map π_S and i_S.
3) Here we use the induction hypothesis. We have the following commutative diagram:

$$\begin{tikzcd}
R \arrow[r] & R_{\mathfrak{q}_S} \arrow[r] \arrow[d] & H_{\mathfrak{p}_S} \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & \overline{K} \\
 & Frac(\frac{R}{\mathfrak{q}_S}) \arrow[r] & Frac(\frac{H}{\mathfrak{p}_S}) \arrow[r] & \frac{S}{\mathfrak{m}_S} &
\end{tikzcd}$$

Observe that $Frac(\frac{H}{\mathfrak{p}_S})$ is an algebraic extension of $Frac(\frac{R}{\mathfrak{q}_S})$ and that $\frac{S}{\mathfrak{m}_S}$ is the separable closure of $Frac(\frac{H}{\mathfrak{p}_S})$, so that it is also the separable closure of $Frac(\frac{R}{\mathfrak{q}_S})$. Moreover the natural map $G_S \to Gal(Frac(\frac{R}{\mathfrak{q}_S}))$ is surjective and hence the two actions on $End_{G_S}(T_l(A_{k_S}))$ have the same fixed point. But the transcendence degree of $Frac(\frac{R}{\mathfrak{q}_S})$ is one less of the one of K, since $\mathfrak{q}_s$ is of height one, so by induction we have that $End_{G_S}(T_l(A_{k_S})) \subseteq End(A_{k_S}) \otimes \mathbb{Z}_l$.
4)We have to show that $EA(r_S)$ is surjective and we start observing that r_S is injective. As in the previous proposition we get the following commutative diagram

$$\begin{tikzcd}
End(A_{\frac{H}{\mathfrak{p}_S}}|\frac{H}{\mathfrak{p}_S})_S \arrow[r] \arrow[d] & End(A_{\frac{H}{\mathfrak{p}_S}}|\frac{H}{\mathfrak{p}_S}) \arrow[d] \\
Spec(k_S) \arrow[r] & Spec(\frac{H}{\mathfrak{p}_S})
\end{tikzcd}$$

An element in $End(A_{k_S})$ is a map $Spec(k_S) \to End(A_{\frac{H}{\mathfrak{p}_S}}|\frac{H}{\mathfrak{p}_S})$ and hence, as in the previous proposition a map $\frac{\frac{H}{\mathfrak{p}_S}}{I} \to k_S$ for some ideal I of $\frac{H}{\mathfrak{p}_S}$. But since r_S is injective we get that $I = 0$ and hence it comes from a section $Spec(\frac{H}{\mathfrak{p}_S}) \to End(A_{\frac{H}{\mathfrak{p}_S}}|\frac{H}{\mathfrak{p}_S})$.

- Step 2. $ET_l(\pi_H)ET_{i_H}^{-1}(End_{\Gamma_K}(T_l(A_{\overline{K}})) \subseteq \cap_{S\in M}(Im(EA(\pi_{\frac{H}{\mathfrak{p}_S}})) \otimes \mathbb{Z}_l)$
From the previous point we get $ET_l(\pi_H^S)ET_l(i_H)^{-1}(End_G(T_l(A_{\overline{K}}))) \subseteq End(A_{\frac{H}{\mathfrak{p}_S}}) \otimes \mathbb{Z}_l$ for every S. Applying $ET_l(\pi_{\frac{H}{\mathfrak{p}_S}})$ we get, thanks to the commutativity of the diagram at the beginning of the proof, $ET_l(\pi_H)ET_l(i_H)^{-1}(End_G(T_l(A_{\overline{K}}))) \subseteq ET_l(\pi_{\frac{H}{\mathfrak{p}_S}})(End(A_{\frac{H}{\mathfrak{p}_S}}) \otimes \mathbb{Z}_l)$. But the last term is included in $Im(EA(\pi_{\frac{H}{\mathfrak{p}_S}}) \otimes \mathbb{Z}_l)$ thanks to the following commutative diagram:

$$\begin{tikzcd}
End(A_{\frac{H}{\mathfrak{p}_S}}) \otimes \mathbb{Z}_l \arrow[r] \arrow[d] & End(A_{k_S}) \otimes \mathbb{Z}_l \arrow[d] \\
End(T_l(A_{\frac{H}{\mathfrak{p}_S}})) \arrow[r] & End(T_l(A_{k_S}))
\end{tikzcd}$$

- Step 3. Conclusion of the proof.
Thanks to the previous point and the lemmas in chapter 3, we get

$$ET_l(\pi_H)ET_l(i_H)^{-1}(End_{\Gamma_K}(T_l(A_{\overline{K}})) \subseteq \cap_{S\in M}(Im(EA(\pi_{\frac{H}{\mathfrak{p}_S}})) \otimes \mathbb{Z}_l) =$$

$$= (\cap_{S\in M} Im(EA(\pi_{\frac{H}{\mathfrak{p}_S}}))) \otimes \mathbb{Z}_l = ImEA(\pi_H) \otimes \mathbb{Z}_l = EA(\pi_H)(End(A_H) \otimes \mathbb{Z}_l)$$

and hence that $ET_l(i_H)^{-1}(End_{\Gamma_K}(T_l(A_{\overline{K}})) \subseteq End(A_H) \otimes \mathbb{Z}_l$. Applying $ET_l(i_H)$ we get $End_{\Gamma_K}(T_l(A_{\overline{K}}) \subseteq End(A_{\overline{k}}) \otimes \mathbb{Z}_l$ so that $End_{\Gamma_K}(T_l(A_{\overline{K}})) = End(A_{\overline{K}}) \otimes \mathbb{Z}_l \cap End_{\Gamma_K}(T_l(A_{\overline{K}}))$. To conclude, just observe that $End(A_{\overline{K}}) \otimes \mathbb{Z}_l \cap End_{\Gamma_K}(T_l(A_{\overline{K}})) = End(A_K) \otimes \mathbb{Z}_l$ since the only morphism that are fixed by the Galois are the one defined over K.

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Index of definitions and notations

Notations

In this thesis we have used the following convention:

- The letters k, K, F usually denote fields and $\overline{k},\overline{K},\overline{F}$ their algebraic closures.
- We denote the absolute Galois group of a field k as Γ_k or as $\pi_1(k)$
- More generally, we denote with $\pi_1(X)$ the étale fundamental group of a connected scheme, implicitly assuming the choice of a base point.
- A, B usually are used for abelian varieties and L, M for line bundles over them.
- The polarizations are denoted with λ or with ψ_L when we want to emphasize that they come from a line bundle L.
- $\mathcal{P}_A$ denotes the Poincaré bundle of an abelian variety A.
- The projections $A \times A \to A$ are denote with p, q or with π_1,π_2.
- G usually denotes a group scheme, the multiplication is denoted with m and the unit section with ϵ.
- The multiplication by n is denoted with $[n]$ and its kernel $G[n]$

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